



COLLEGE PARK CAMPUS



**On the Regularity of Elasticity Problems
with Piecewise Analytic Data**

by

B. Q. Guo

and

I. Babuška

Technical Note BN-1136

93-01608 *no*



5686

July 1992



**INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY**

DISTRIBUTION STATEMENT A

Approved for public release
Distribution Unlimited

On the Regularity of Elasticity Problems
with Piecewise Analytic Data

B. Q. Guo¹
Department of Applied Mathematics
University of Manitoba, Winnipeg MB
R3T 2N2, Canada

I. Babuška²
Institute for Physical Science and Technology
University of Maryland,
College Park, MD 20742

DTIC QUALITY INSPECTED 3

MD-92-14-GB
TR92-14

April 1992

Accession For	
NTIS ANAL	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By Res 9254767	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

¹Partially supported by the National Science and Engineering Research Council of Canada, under Grant OGP00 48726.

²Partially supported by the Office of Naval Research under Grant N00014-90-J1030.

ABSTRACT

This paper analyzes the regularity of plane elasticity problems with piecewise analytic data. The results are given in the frames of the countably weighted Sobolev spaces introduced by Babuška and Guo [SIAM J. Math. Anal., Vol 19 (1988), pp. 172-203, Vol. 20 (1989), pp. 763-781]. These results are of major significance for the practical design and theoretical analysis of the p and h - p version of the finite element solutions of the elasticity problems.

Key Words. Elasticity problem with piecewise analytic data, countably normed space, weighted Sobolev space, singularity.

AMS(MOS) Subject Classifications. 35B65, 35D10, 35G15, 35J05, 73C02

§1. INTRODUCTION

The regularity of the boundary value problems for elliptic equations was addressed in many papers. The regularity of the solution, when the domain is not smooth, was addressed in [18,19] and subsequently in many papers (see e.g. [13,20,21,22,26]). The analysis was made for the scalar equation (e.g. Laplace equation) and for general elliptic systems, e.g. theory of elasticity. Although the results are essentially similar, the proofs are in various details different. The regularity of the solution is typically described in terms of Sobolev spaces and various decompositions of the solution in the singular and regular parts. These results are important not only for theoretical reasons but also for the design and analysis of the numerical treatment of these problems.

Finite element method is today the mostly used method for solving the elliptic equations. There are three versions of the finite element method, the h , p and h - p finite element method (see e.g. [7]). The following questions arise in the connection with the FEM:

- a) What is a typical class of problems in applications, e.g. structural mechanics,
- b) What regularity results are the most useful for the design and analysis of the finite element method, especially its h - p version.

The answer to the first question is that in the structural mechanics we deal with the problems with piecewise analytic data (boundary of the domain, coefficients of the equation and boundary conditions, etc.). Description by the standard Sobolev of finite order neglects features important for the numerical solution.

The answer to the second question is that we need such a description which allows to construct the numerical method which is maximally effective.

In the case of the p -version of the finite element the user provides the mesh, and the degree is adaptively selected (see e.g., Applied Structure (Rasna Corp. CA, USA) or STRIPE (Aeronautical Research Inst. of Sweden)). The proper regularity description leads to the rules which the user has to follow (see e.g. [27]). In the case of the h - p version the proper form of the regularity results allow to construct the meshes and degrees such that the convergence of the method is exponential, see e.g. [4,5,6,14].

We have found that the proper form of the regularity results which serves best the goals of the numerical analysis is the description of the regularity in the frame of countably normed (weighted) Sobolev spaces. Although the spaces are close to the Gevrey spaces (see [9,11]) they have some special different features, and the form of the statements are directly related to the effective use in the FEM. We have introduced these spaces in [2,3,4] and shown their significant applications for the finite element analysis. We have proven that the h - p version with an exponential rate of convergence can be constructed.

In [2,3,4] we addressed the problem of the scalar equation. Here we address the elasticity problem. Although the basic results and the analysis methodology are similar here and for the scalar problem, there are various essential details which need significant modification of the analysis. Because of the major importance of the elasticity in application, the detailed theoretical results presented here are essential. They do not follow directly or easily from the available results in the literature.

In Section 2 we introduce the weighted Sobolev spaces and countably normed space. The variational solution of the elasticity problem with data given in weighted spaces is addressed in Section 3. Section 4 gives a complete analysis of the elasticity problem in an infinite angular domain.

In Section 5 we prove the regularity theorems for the elasticity problem in polygonal domains in terms of the countably weighted Sobolev spaces.

The methodology and techniques used for the elasticity problem can be used for general elliptic systems of equations.

§2. PRELIMINARIES

The notations and definition of spaces in [14] are the ones most often used in this paper. Let Ω denote a polygon with vertices A_i and open edges Γ_i connecting A_i and A_{i+1} , $1 \leq i \leq M$ ($A_{M+1} = A_1$) shown in Fig. 2.1. Let \mathcal{D} and \mathcal{N} be subsets of $M = \{1, 2, \dots, M\}$, $\mathcal{D} = M \setminus \mathcal{N}$, $\Gamma^0 = \bigcup_{i \in \mathcal{D}} \Gamma_i$ and $\Gamma^1 = \bigcup_{i \in \mathcal{N}} \Gamma_i$, $\Gamma = \partial\Omega = \Gamma^0 \cup \Gamma^1$. Let $\beta = (\beta_1, \dots, \beta_M)$ be M -tuple of real numbers, $0 < \beta_i < 1$, $1 \leq i \leq M$, and $\Phi_{\beta+k} = \prod_{i=1}^M r_i^{\beta_i+k}(x)$ where k is an integer and r_i is the distance between x and vertex A_i .

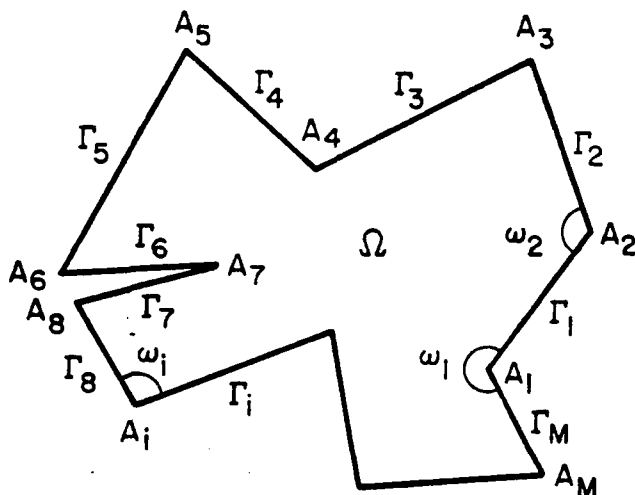


Fig. 2.1 Polygonal Domain

By $H^k(\Omega)$ we denote the usual Sobolev spaces and by $H_{\beta}^{k,\ell}(\Omega)$ the weighted Sobolev spaces for integer k , ℓ , $k \geq \ell \geq 0$ with the norm

$$\|u\|_{H_{\beta}^{k,\ell}(\Omega)}^2 = \|u\|_{H^{\ell-1}(\Omega)}^2 + \sum_{|\alpha| \geq \ell} \|\Phi_{\beta+|\alpha|-\ell} D^{\alpha} u\|_{L^2(\Omega)}^2$$

(if $\ell = 0$, the term $\|u\|_{H^{\ell-1}(\Omega)}^2$ drops out) where $D^{\alpha} u = u \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$,

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geq 0$ integer, $i = 1, 2$. We shall write $H_{\beta}^{0,0}(\Omega) = L_{\beta}(\Omega)$.

For convenience, we use the spaces $\mathcal{H}_\beta^{k,\ell}(Q)$ instead of $H_\beta^{k,\ell}(Q)$ if Q is finite or infinite sector, with the norm

$$\|u\|_{\mathcal{H}_\beta^{k,\ell}(Q)}^2 = \|u\|_{H^{\ell-1}(Q)}^2 + \|r^{|\alpha| - \ell + \beta} \mathcal{D}^\alpha u\|_{L^2(Q)}^2$$

(if $\ell = 0$, the term $\|u\|_{H^{\ell-1}(Q)}^2$ drops out) where $\mathcal{D}^\alpha u = u \frac{\partial^{|\alpha|}}{\partial r^{\alpha_1} \partial \theta^{\alpha_2}} = \frac{\partial^{|\alpha|} u}{\partial r^{\alpha_1} \partial \theta^{\alpha_2}}$,

(r, θ) is the polar coordinate, α is the same as above.

The spaces $\mathcal{W}_\beta^k(Q)$ introduced by Kondratév (see [16, 17, 18]), are used in this paper as well in polar coordinates, and

$$\|u\|_{\mathcal{W}_\beta^k(Q)}^2 = \sum_{|\alpha| \leq k} \int_Q r^{2(\beta - k + \alpha_1)} |\mathcal{D}^\alpha u|^2 r dr d\theta.$$

Let $D = \{(\tau, \theta) |, -\infty < \tau < \infty, 0 < \theta < \omega\}$, and for integer $k \geq 0$ and real $h > 0$ we define

$$H_h^k(D) = \left\{ u \mid \sum_{0 \leq |\alpha| \leq k} \int_D e^{2h\tau} |\mathcal{D}^\alpha u|^2 d\tau d\theta = \|u\|_{H_h^k(D)}^2 < \infty \right\}.$$

We shall write $H_h^0(D) = L_h(D)$.

The countably normed spaces $B_\beta^\ell(\Omega)$, $\ell = 0, 1, 2$, are defined by

$$B_\beta^\ell(\Omega) = \{u \mid u \in H_\beta^{k,\ell}(\omega), \|\Phi_{\beta+k-\ell} \mathcal{D}^\alpha u\|_{L_2(\omega)} \leq C d^{k-\ell} (k-\ell)!,$$

for $|\alpha| = k = \ell, \ell+1, \dots$, with $C, d \geq 1$ independent of k ,

and $\mathcal{B}_\beta^\ell(S)$ is defined in polar coordinates on finite sector $S = \{(r, \theta) \mid 0 < r < \delta, 0 < \theta < \omega\}$ by

$$\mathcal{B}_\beta^\ell(S) = \{u \mid u \in \mathcal{H}_\beta^{k,\ell}(S), \|r^{\beta-\ell+\alpha_1} \mathcal{D}^\alpha u\|_{L_2(S)} \leq C d^{k-\ell} (k-\ell)!,$$

for $|\alpha| = k = \ell, \ell+1, \dots$, with $C, d \geq 1$, independent of k .

We shall write $B_\beta^\ell(\Omega, C, d)$ or $\mathcal{B}_\beta^\ell(S, C, d)$ if we emphasize the constants C and d .

The spaces $H_{\beta}^{k-1/2, \ell-1/2}(\Gamma^m)$ and $B_{\beta}^{\ell-1/2}(\Gamma^m)$ are defined as the trace spaces of $H_{\beta}^{k, \ell}(\Omega)$ and $B_{\beta}^{\ell}(\Omega)$ on Γ^m , $m = 0, 1$, $\ell = 1, 2$, and

$$\|g\|_{H_{\beta}^{k-1/2, \ell-1/2}(\Gamma^m)} = \inf_{G|_{\Gamma^m}=g} \|G\|_{H_{\beta}^{k, \ell}(\Omega)}.$$

We shall use the notation $D^k u$ and $\mathcal{D}^k u$ which are defined as

$$|D^k u|^2 = \sum_{|\alpha|=k} |D^{\alpha} u|^2$$

and

$$|\mathcal{D}^k u|^2 = \sum_{|\alpha|=k} r^{-2\alpha_2} |\mathcal{D}^{\alpha} u|^2$$

The theorem on the equivalence of weighted Sobolev spaces defined in Cartesian coordinates and polar coordinates will be used later. We quote the following theorem from [2].

Theorem 2.1 (cf. Theorem 2.1 of [2]). Let S be a finite sector $= \{(r, \theta) | 0 < r < \delta, 0 < \theta < \omega\}$, $\Phi_{\beta} = r^{\beta}$, $0 < \beta < 1$, then for $0 \leq \ell \leq 2$ integer, $u \in H_{\beta}^{k, \ell}(S)$, (resp. $B_{\beta}^{\ell}(S)$) if and only if $u \in \mathcal{H}_{\beta}^{k, \ell}(S)$ (resp. $\mathcal{B}_{\beta}^{\ell}(S)$). \square

We denote the vector and vector space by bold face. For example, $u = (u_1, u_2)^T$, $H^k(\Omega) = H^k(\Omega) \times H^k(\Omega)$, $B_{\beta}^{\ell}(\Omega) = B_{\beta}^{\ell}(\Omega) \times B_{\beta}^{\ell}(\Omega)$, etc. The theorem quoted above holds for vector spaces $H_{\beta}^{k, \ell}(S)$ and $\mathcal{H}_{\beta}^{k, \ell}(S)$ (resp. $B_{\beta}^{\ell}(\Omega)$ and $\mathcal{B}_{\beta}^{\ell}(\Omega)$) as well.

§3. VARIATIONAL SOLUTION OF ELASTICITY PROBLEM WITH
DATA GIVEN IN WEIGHTED SPACES

Consider the linear plain strain elasticity problem on polygon Ω

$$(3.1a) \quad \begin{cases} -\mu \Delta u_1 - (\lambda + \mu) \frac{\partial}{\partial x_1} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = f_1 \\ -\mu \Delta u_2 - (\lambda + \mu) \frac{\partial}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = f_2 \end{cases} \quad \text{in } \Omega$$

$$(3.1b) \quad \begin{cases} u|_{\Gamma^0} = g^0 = G^0|_{\Gamma^0} \\ T = \sigma \cdot n|_{\Gamma^1} = g^1 = G^1|_{\Gamma^1} \end{cases}$$

where $u = (u_1, u_2)^T$ is displacement, and $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ is the stress tensor given by

$$(3.2) \quad \begin{cases} \sigma_{11} = 2\mu \frac{\partial u_1}{\partial x_1} + \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \\ \sigma_{22} = 2\mu \frac{\partial u_2}{\partial x_2} + \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \\ \sigma_{12} = \sigma_{21} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \end{cases}$$

λ and μ are Lamé coefficients, n is unit outward normal to $\partial\Omega$, the force $f = (f_1, f_2)^T$, the boundary condition $g^\ell = (g_1^\ell, g_2^\ell)^T$, $G^\ell = (G_1^\ell, G_2^\ell)^T$, etc.

Using the differential operator L and the boundary operator B we write (3.1) as

$$(3.1') \quad \begin{cases} Lu = f & \text{in } \Omega \\ Bu|_{\Gamma} = [g^0, g^1] \end{cases}$$

Theorem 3.1. Let $f \in L_\beta(\Omega) = H_\beta^{0,0}(\Omega)$, $g^\ell \in H_\beta^{3/2-\ell, 3/2-\ell}(\Gamma^\ell)$, $\ell = 0, 1$, and $|\Gamma^0| \neq 0$, then problem (3.1) has a unique solution $u \in H^1(\Omega)$ (in the weak

sense), and

$$(3.3) \quad \|u\|_{H^1(\Omega)} \leq C \left[\|f\|_{L_\beta(\Omega)} + \sum_{\ell=0,1} \|g^\ell\|_{H_\beta^{3/2-\ell, 3/2-\ell}(\Gamma^\ell)} \right].$$

Proof. First we assume that $g^0 = 0$. The bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ is

$$(3.4) \quad B(u, v) = \int_{\Omega} \left\{ 2\mu \left(\frac{\partial u_1}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) + \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \right. \\ \left. + \mu \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) \right\} dx dy$$

where $H_0^1(\Omega) = \{u | u \in H^1(\Omega), u|_{\Gamma^0} = 0\}$.

Owing to Lemma 2.9 and 2.11 of [2] we have

$$|\int_{\Gamma^1} g^1 \cdot v ds| \leq C \|g^1\|_{H_\beta^{1/2, 1/2}(\Gamma^1)} \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

From Lemma 2.10 of [2] we have $f \in (H^1(\Omega))'$, and

$$|\int_{\Omega} f \cdot v dx| \leq C' \|f\|_{L_\beta(\Omega)} \|v\|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

Hence

$$F(v) = \int_{\Omega} f \cdot v dx + \int_{\Gamma^1} g^1 \cdot v ds$$

is a linear and continuous functional on $H^1(\Omega)$, and

$$\|F\|_{(H^1(\Omega))'} \leq C \left[\|f\|_{L_\beta(\Omega)} + \|g^1\|_{H_\beta^{1/2, 1/2}(\Gamma^1)} \right].$$

The variational problem is to seek $u \in H_0^1(\Omega)$ such that

$$B(u, v) = F(v) \quad \text{for any } v \in H_0^1(\Omega).$$

There is a well-known Korn inequality, i.e., for $u \in H_0^1(\Omega)$, $|\Gamma^0| \neq 0$,

$$C_1 \|u\|_{H^1(\Omega)}^2 \leq \int_{\Omega} \left\{ \left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_2} \right)^2 + \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 \right\} dx \leq C_2 \|u\|_{H^1(\Omega)}^2.$$

where $C_1 > 0$ ($i = 1, 2$) are independent of u (see [17]). Applying the Lax-Milgram theorem, we have existence and uniqueness of the weak solution of (3.1), and (3.3) holds. If $g^0 \neq 0$, let $w = u - G^0$, then $w \in H_0^1(\Omega)$ satisfies

$$\begin{aligned} B(w, v) &= \int_{\Omega} f \cdot v dx + \int_{\Gamma^1} \left[g^1 - \frac{\partial G^0}{\partial n} \right] \cdot v ds - B(G^0, v), \quad \forall v \in H_0^1(\Omega) \\ &= \tilde{F}(v). \end{aligned}$$

Obviously $\tilde{F} \in (H^1(\Omega))'$. Applying the result above we obtain the existence and uniqueness of the solution of (3.1) in general (in the weak sense).

§4. AUXILIARY PROBLEMS ON INFINITE SECTOR

In order to analyze the behavior of the solution at the corners of Ω , we consider the elasticity equation (3.1) on an infinite sector $Q = \{(r, \theta) | 0 < r < \infty, 0 < \theta < \omega\}$. We prefer to write the equation in polar coordinates (see e.g. [25]). Let $\bar{u} = (u_r, u_\theta)^T$ and $\bar{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{\theta r} & \sigma_{\theta\theta} \end{bmatrix}$, $\bar{f} = (f_r, f_\theta)^T$, etc., and let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Then $\bar{u} = Au$, $\bar{f} = Af$, $\bar{g}^\ell = Ag^\ell$, $\bar{\sigma} = A\sigma A^T$. \bar{u} satisfies

$$\begin{aligned} (4.1) \quad & -\mu \left(\Delta u_r - \frac{1}{r^2} u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) - (\lambda + \mu) \frac{\partial}{\partial r} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) = f_r \\ & -\mu \left(\Delta u_\theta - \frac{1}{r^2} u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) - (\lambda + \mu) \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) = f_\theta \end{aligned} \quad \text{in } Q,$$

and $\bar{\sigma}$ is given by

$$\sigma_{rr} = 2\mu \frac{\partial u_r}{\partial r} + \lambda \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right)$$

$$(4.2) \quad \sigma_{\theta\theta} = 2\mu \left[\frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r} \right] + \lambda \left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} \right]$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu \left[\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right].$$

One of three kinds of boundary conditions may be imposed on ∂Q , namely

$$(4.3a) \quad \bar{u} |_{\theta=0, \omega} = \bar{g}^0 = \bar{G}^0 |_{\sigma=0, \omega} \quad (\text{Dirichlet condition})$$

$$(4.3b) \quad \bar{T} |_{\theta=0, \omega} = (\sigma_{r\theta}, \sigma_{\theta\theta})^T = \bar{g}^1 = \bar{G}^1 |_{\theta=0, \omega} \quad (\text{Neumann condition})$$

$$(4.3c) \quad \bar{u} |_{\theta=0} = \bar{g}_0^0 = \bar{G}^0 |_{\theta=0}, \quad (\sigma_{r\theta}, \sigma_{\theta\theta})^T |_{\theta=\omega} = \bar{g}_{\omega}^1 = \bar{G}^1 |_{\theta=\omega} \quad (\text{Mixed condition}).$$

Hereafter, \bar{g}^l denotes the different vectors $\bar{g}_0^l = \bar{G}^l(r, 0)$ and $\bar{g}_{\omega}^l = \bar{G}^l(r, \omega)$

in the trace sense at $\theta = 0, \omega$. Let $\bar{L}(\mathcal{D})$ and $\bar{B}(\mathcal{D})$ be corresponding matrix-differential operator and boundary operator. We rewrite (4.1) and (4.3) as

$$(4.4) \quad [\bar{L}(\mathcal{D}), \bar{B}(\mathcal{D})] \bar{u} = [\bar{f}, \bar{g}^0, \bar{g}^1].$$

By the change of variable $\tau = \ln \frac{1}{r}$, we convert the equation (4.1) and boundary condition (4.3) into the problem $\tilde{L}(\tilde{u}) = \tilde{f}$ on a strip domain $D = \{(\tau, \theta) \mid -\infty < \tau < \infty, 0 < \theta < \omega\}$, i.e.,

$$(4.5) \quad \begin{cases} -(2\mu + \lambda) \left[\frac{\partial^2 u_{\tau}}{\partial \tau^2} - \tilde{u}_{\tau} \right] - \mu \frac{\partial^2 u_{\tau}}{\partial \theta^2} + (\mu + \lambda) \frac{\partial^2 u_{\theta}}{\partial \tau \partial \theta} + (3\mu + \lambda) \frac{\partial \tilde{u}_{\theta}}{\partial \theta} = \tilde{f}_{\tau} \\ -(\mu + \lambda) \frac{\partial^2 \tilde{u}_{\tau}}{\partial \tau \partial \theta} - (3\mu + \lambda) \frac{\partial \tilde{u}_{\tau}}{\partial \theta} - \mu \frac{\partial^2 \tilde{u}_{\theta}}{\partial \tau^2} - (2\mu + \lambda) \frac{\partial^2 \tilde{u}_{\theta}}{\partial \theta} + \mu \tilde{u}_{\theta} = \tilde{f}_{\theta} \end{cases} \quad \text{in } D$$

with one of the following boundary conditions:

$$(4.6a) \quad \tilde{u}|_{\theta=0, \omega} = \tilde{g}^0 = \tilde{G}^0|_{\theta=0, \omega}, \quad (\text{Dirichlet condition})$$

$$(4.6b) \quad \tilde{T} = (\tilde{\sigma}_{\tau\theta}, \tilde{\sigma}_{\theta\theta})^T|_{\theta=0, \omega} = \tilde{g}^1 = \tilde{G}^1|_{\theta=0, \omega}, \quad (\text{Neumann condition})$$

$$(4.6c) \quad \tilde{u}|_{\theta=0} = \tilde{g}_0^0 = \tilde{G}^0|_{\theta=0}, \quad \tilde{T} = (\tilde{\sigma}_{\tau\theta}, \tilde{\sigma}_{\theta\theta})^T|_{\theta=\omega} = \tilde{g}_\omega^1 = \tilde{G}^1|_{\theta=\omega} \\ (\text{Mixed condition})$$

where $\tilde{u} = (\tilde{u}_\tau, \tilde{u}_\theta)^T = (u_\tau(e^{-\tau}, \theta), u_\theta(e^{-\tau}, \theta))^T$, $\tilde{G}^\ell = (G_\tau^\ell, G_\theta^\ell)^T = e^{-\ell\tau}(G_\tau^\ell(e^{-\tau}, \theta), G_\theta^\ell(e^{-\tau}, \theta))^T$, $\ell = 0, 1$, $\tilde{f} = (\tilde{f}_\tau, \tilde{f}_\theta)^T = e^{-2\tau}(f_\tau(e^{-\tau}, \theta), f_\theta(e^{-\tau}, \theta))^T$, and

$$(4.7) \quad \tilde{\sigma}_{\tau\theta} = \mu \left[\frac{\partial \tilde{u}_\tau}{\partial \theta} - \frac{\partial \tilde{u}_\theta}{\partial \tau} - \tilde{u}_\theta \right], \quad \tilde{\sigma}_{\theta\theta} = 2\mu \left[\frac{\partial \tilde{u}_\theta}{\partial \theta} + \tilde{u}_\tau \right] + \lambda \left[-\frac{\partial \tilde{u}_\tau}{\partial \tau} + \tilde{u}_\tau + \frac{\partial \tilde{u}_\theta}{\partial \theta} \right].$$

Further, by Fourier transformation we obtain a system of ordinary equations

$$(4.8) \quad \begin{cases} -\mu \frac{d^2 \hat{u}_\tau}{d\theta^2} + (2\mu + \lambda)(1 + \eta^2) \hat{u}_\tau + (i\eta(\mu + \lambda) + (3\mu + \lambda)) \frac{d\hat{u}_\theta}{d\theta} = \hat{f}_\tau \\ -((3\mu + \lambda) - i\eta(\mu + \lambda)) \frac{d\hat{u}_\tau}{d\theta} - (2\mu + \lambda) \frac{d^2 \hat{u}_\theta}{d\theta^2} + \mu(1 + \eta^2) \hat{u}_\theta = \hat{f}_\theta \end{cases}$$

with one of the following boundary conditions:

$$(4.9a) \quad \hat{u}|_{\theta=0, \omega} = \hat{g}^0 = \hat{G}^0|_{\theta=0, \omega}, \quad (\text{Dirichlet condition})$$

$$(4.9b) \quad \hat{T}|_{\theta=0, \omega} = (\hat{\sigma}_{\tau\theta}, \hat{\sigma}_{\theta\theta})^T|_{\theta=0, \omega} = \hat{g}^1 = \hat{G}^1|_{\theta=0, \omega}, \quad (\text{Neumann condition})$$

$$(4.9c) \quad \hat{u}|_{\theta=0} = \hat{g}_0^0 = \hat{G}^0|_{\theta=0}, \quad \hat{T} = (\hat{\sigma}_{\tau\theta}, \hat{\sigma}_{\theta\theta})^T|_{\theta=\omega} = \hat{g}_\omega^1 = \hat{G}^1|_{\theta=\omega} \\ (\text{Mixed condition})$$

where $\hat{u} = (\hat{u}_\tau, \hat{u}_\theta)^T = (\mathcal{F}(\tilde{u}_\tau), \mathcal{F}(\tilde{u}_\theta))^T = \mathcal{F}(\tilde{u})$, etc.,

$$\mathcal{F}(\bar{u}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta\tau} \bar{u}(\tau, \theta) d\tau, \quad \eta = \xi + ih, \quad -\infty < \xi < \infty, \quad h > 0,$$

and

$$(4.10) \quad \hat{\sigma}_{\tau\theta} = \mu \left[\frac{d\hat{u}_{\tau}}{d\theta} - (1\eta+1)\hat{u}_{\theta} \right], \quad \hat{\sigma}_{\theta\theta} = (2\mu+\lambda) \frac{d\hat{u}_{\theta}}{d\theta} + (2\mu+\lambda-i\lambda\eta)\hat{u}_{\tau}.$$

By $\hat{L}(D_{\theta}, \eta)$ and $\hat{B}(D_{\theta}, \eta)$ we denote the matrix-differential operator and boundary operator. Furthermore, we write the system and the boundary condition as

$$(4.11) \quad \mathcal{U}(\eta)\hat{u} = [\hat{L}(D_{\theta}, \eta), \hat{B}(D_{\theta}, \eta)]\hat{u} = [\hat{f}, \hat{G}^0, \hat{G}^1] \quad (\text{or } [\hat{f}, \hat{g}^0, \hat{g}^1].)$$

The operator $\mathcal{U}(\eta) = [\hat{L}(D_{\theta}, \eta), \hat{B}(D_{\theta}, \eta)]$ depends polynomially on the complex parameter η . By the argument used in [14] for all η , with the exception of certain isolated points, $\mathcal{U}(\eta)$ realizes an isomorphism:

$H^2(I) \approx L^2(I) \times H^2(I) \times H^1(I)$ (or $L^2(I) \times C^2 \times C^2$). Consequently, the inverse operator $\mathcal{R}(\eta) = \mathcal{U}(\eta)^{-1}$ is an operator-valued meromorphic function of η with poles of finite multiplicity. These poles are the eigenvalues of $\mathcal{U}(\eta)$ (see [10, 16, 18, 19, 20, 21, 22]). For each pole η of $\mathcal{R}(\eta)$, the homogeneous problem of (4.11) has at least one non-trivial solution corresponding (eigenvector function) in $H^2(I)$. The transcendental equations which the eigenvalues satisfy have been derived in several different ways and can be seen in the literatures of continuum mechanics and mathematics (see e.g. [10, 24, 26, 28]). The typical approach is to consider a biharmonic equation in Q instead of the elasticity equations. Since we adopt the displacement $\bar{u} = (u_r, u_{\theta})^T$ in polar coordinates in (4.8) and (4.9) the coefficients of the operator $\hat{L}(D_{\theta}, \eta)$ and $\hat{B}(D_{\theta}, \eta)$ are constants. Hence we are able to derive the transcendental equations directly from the homogeneous equation (4.8) and the boundary condition (4.9) on displacement and traction.

Since these equations are not of the main interest in this paper we will present them in a lemma below. For completeness we include the proof in the Appendix.

Lemma 4.1. Let $\eta = iz$ be an eigenvalue of $\mathcal{U}(\eta)$, and ν be the Poisson ratio. Then

(i) for the Dirichlet problem (4.8) and (4.9a), z satisfies

$$(4.12) \quad \sin^2 z\omega = \left[\frac{z}{3-4\nu} \right]^2 \sin^2 \omega,$$

(ii) for the Neumann problem (4.8) and (4.9b), z satisfies

$$(4.13) \quad \sin^2 z\omega = z^2 \sin^2 \omega,$$

specially $\eta = 0$ is an eigenvalue with multiplicity of 2, the corresponding eigenvectors are $\bar{e}_1 = (\cos \theta, -\sin \theta)^T$ and $\bar{e}_2 = (\sin \theta, \cos \theta)^T$,

(iii) for the mixed problem (4.8) and (4.9c), z satisfies

$$(4.14) \quad \sin^2 z\omega = \frac{4(1-\nu)^2}{3-4\nu} - \frac{z^2}{3-4\nu} \sin^2 \omega$$

and $\eta = \pm i$ (i.e., $z = \pm 1$) are the eigenvalues with multiplicity of 1 if $(1 + \frac{\lambda}{\mu}) \cos 2\omega + 1 = 0$. □

From the equation (4.11) ~ (4.13) it is easily seen that zeroes of these equations are symmetric with respect to the origin and the real axis in complex plane. Hence the eigenvalues of $\mathcal{U}(\eta)$ are located in the complex plane symmetrically with respect to the origin and the imaginary axis. By T_η we denote the eigenvalues, and let κ_1 be a positive number such that

$$(4.15) \quad \kappa_1 = \min_{\substack{\eta \in T_\eta \\ \text{Im } \eta > 0}} \text{Im } \eta = \min_{\substack{\eta \in T_\eta \\ \text{Im } \eta \neq 0}} |\text{Im } \eta|.$$

Next we prove the Agranovich & Vishik conditions I and II which are substantial to the key inequality (4.17) (see [1]). These two conditions were

used implicitly or explicitly in many papers, e.g., [26] for elasticity problems and in [21,22] for general elliptic systems.

Let $\hat{D}u = i \frac{du}{d\theta}$ and $\hat{A}_0(D, \eta)$ be the principal part of the operator $\hat{L}(D_\theta, \eta)$. We write $\hat{A}_0(D, \eta)$ in matrix form

$$\hat{A}_0(D, \eta) = \begin{bmatrix} \eta^2(2\mu+\lambda) + \mu D^2 & (\mu+\lambda)\eta D \\ (\mu+\lambda)\eta D & \mu\eta^2 + (2\mu+\lambda)D^2 \end{bmatrix}.$$

Lemma 4.2. (Condition I) For $\xi \in \mathbb{R}^1$ (real), $\eta \in \Sigma_{\phi_1} = \{\eta \mid |\arg \eta| < \phi_1 \text{ or } |\arg \eta - \pi| < \phi_1\}$ with any $\phi_1 \in (0, \pi/2)$ and $|\eta| + |\xi| \neq 0$, $\det(\hat{A}_0(\xi, \eta)) \neq 0$. Furthermore, the equation $\det(\hat{A}_0(\zeta, \eta)) = 0$ in ζ has equal numbers of roots in upper and lower half-planes for $\eta \in \Sigma_{\phi_1}$ and $\eta \neq 0$.

Proof. It is easy to see that

$$\det(\hat{A}_0(\xi, \eta)) = \mu(2\mu+\lambda)(\eta^2 + \xi^2)^2 \neq 0$$

for $\xi \in \mathbb{R}^1$, $\eta \in \Sigma_{\phi_1}$ with any $\phi_1 \in (0, \pi/2)$, and $|\xi| + |\eta| \neq 0$. Also it is seen that $\zeta = \pm i\eta$ are the roots of the equation $\det(\hat{A}_0(\zeta, \eta)) = 0$ in ζ (complex). Hence the equation has 2 roots in upper and lower plane, respectively if $0 \neq \eta \in \Sigma_{\phi_1}$. □

Let $\hat{B}_0(D, \eta)$ be the principal part of the boundary operator $\hat{B}(D_\theta, \eta)$ defined by (4.10), and $\theta_0 = 0$ (resp. ω), $I_\theta = (0, \omega)$ (resp. $I_\theta = (-\omega, \omega)$). Then we have the following lemma.

Lemma 4.3. (Condition II) For any $\phi_1 \in (0, \pi/2)$, if $\eta \neq 0$ and $\eta \in \Sigma_{\phi_1} = \{\eta \mid |\arg \eta - \pi| < \phi_1 \text{ or } |\arg \eta| < \phi_1\}$, the equation on the half line

$$\begin{aligned} \hat{A}_0(D, \eta)\hat{W} &= 0, \quad \theta \in I_{\theta_0} \\ (4.16) \quad \hat{B}_0(D, \eta)\hat{W}|_{\theta=\theta_0} &= \hat{h} \end{aligned}$$

has a unique stable solution \hat{W} such that $|\hat{W}| \rightarrow 0$ as $\theta \rightarrow \infty$ (resp. $\theta \rightarrow -\infty$).

Proof. We will prove the lemma for $\theta_0 = 0$ and $I_{\theta_0} = (0, \infty)$. The proof for $\theta_0 = \omega$ is similar with what follows. For the homogeneous equation (4.16) the solution \hat{W} must have the form $e^{b\theta} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ with b satisfying the equation

$$\det(\hat{A}_0(ib, \eta)) = \mu(2\mu + \lambda)(b^2 - \eta^2)^2 = 0.$$

Hence $b = \pm\eta$ are the roots with multiplicity of 2. For $\eta \in \Sigma_{\phi_1}$ and $\eta \neq 0$, $\operatorname{Re} b = \operatorname{Re} \eta \neq 0$. Let $\alpha = -\operatorname{sgn}(\operatorname{Re} \eta)$, $\sigma = \frac{\lambda + 3\mu}{\lambda + \mu}$

$$\hat{W} = \begin{pmatrix} W_r \\ W_\theta \end{pmatrix} = c_1 e^{\alpha\eta\theta} \begin{pmatrix} 1 \\ \alpha i \end{pmatrix} + c_2 e^{\alpha\eta\theta} \begin{pmatrix} \theta + \alpha\sigma/\eta \\ \alpha i\theta \end{pmatrix}$$

is the stable solution if c_1 and c_2 can be uniquely determined by any given boundary condition $\hat{h} \in \mathbb{C}^2$.

For the Dirichlet condition $\hat{W}|_{\theta=\theta} = \hat{h}$, c_1 and c_2 satisfy

$$\begin{pmatrix} 1 & \alpha\sigma/\eta \\ \alpha i & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \hat{h}.$$

Then obviously c_1 and c_2 can be uniquely solved from the equation above.

For the Neumann condition we have by (4.10)

$$\hat{B}_0(D, \eta)\hat{W} = \begin{pmatrix} \mu \frac{d\hat{W}_r}{d\theta} - i\mu\eta\hat{W}_\theta \\ -i\lambda\eta\hat{W}_r + (2\mu + \lambda) \frac{d\hat{W}_\theta}{d\theta} \end{pmatrix}.$$

Then the Neumann boundary condition $\hat{B}(D, \eta)\hat{W} = \hat{h}$ leads to

$$\begin{pmatrix} 2\mu\eta & \mu(1+\sigma) \\ i2\mu\eta & i(2\mu + \lambda(1-\sigma)) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \hat{h}.$$

Since

$$\det \begin{pmatrix} 2\mu\eta & \mu(1+\sigma) \\ i2\mu\eta & i(2\mu+\lambda(1-\sigma)) \end{pmatrix} = i2\mu\eta (\lambda+\mu) (1-\sigma) = -i4\mu^2\eta \neq 0$$

for $\eta \neq 0$, c_1 and c_2 can be uniquely determined by \hat{h} . Therefore (4.16) has unique stable solution for $\eta \in \Sigma_{\phi_1}$ and $\eta \neq 0$ and

$$|\hat{W}| \leq C_0 e^{-b_0\theta} \rightarrow 0 \text{ as } \theta \rightarrow \infty$$

with $C_0 > 0$ and $b_0 = -|\operatorname{Re} \eta|$.

After verifying the conditions I and II we have the following theorem.

Theorem 4.1. Suppose there is no pole of $\mathcal{R}(\eta)$ on the line $\operatorname{Im} \eta = b$, then the solution $\hat{u} = \mathcal{R}(\eta)[\hat{f}, \hat{G}^0, \hat{G}^1]$ of the problem (4.8) and (4.9) satisfies

$$(4.17) \quad \|\hat{u}\|_{H^2(I)}^2 + |\eta|^4 \|\hat{u}\|_{L^2(I)}^2 \leq C \left[\|\hat{f}\|_{L^2(I)}^2 + \sum_{\ell=0,1} \|\hat{G}^\ell\|_{H^{2-\ell}(I)}^2 + \sum_{\ell=0,1} |\eta|^{3-2\ell} (|\hat{g}_0^\ell|^2 + |\hat{g}_\omega^\ell|^2) \right].$$

Proof. Due to Lemmas 4.1 and 4.2 the conditions I and II are satisfied on an angle $\Sigma_1 = \{\eta: |\arg \eta| < \phi_1 \text{ or } |\arg \eta - \pi| < \phi_1\}$ with $\phi_1 \in (0, \pi/2)$. By Theorem 6.1 of [1], (4.17) holds with C independent of η and \hat{u} if $\eta \in \Sigma_1$ and $|\eta| > \eta_0$, where η_0 is some positive real constant. The line $\operatorname{Im} \eta = b$ is contained in Σ_1 except a finite segment for which $|\operatorname{Re} \eta| < |b| \operatorname{cosec} \phi_1$. Hence (4.17) holds for η on the line $\operatorname{Im} \eta = b$ with $|\eta| > \eta_0$. For those η on the line with $|\eta| \leq \eta_0$, $\mathcal{R}(\eta)$ is analytic. Hence for $\hat{f} \in L^2(I)$, $g^\ell \in C^2$, the solution \bar{u} exists in $H^2(I)$, and

$$\|\hat{u}\|_{H^2(I)}^2 + |\eta|^4 \|\hat{u}\|_{L^2(I)}^2 \leq C_2 (1+|\eta|^4) (\|\hat{f}\|_{L^2(I)}^2 + \sum_{\ell=0,1} (|\hat{g}_0^\ell|^2 + |\hat{g}_\omega^\ell|^2))$$

$$\leq C_3(\|\hat{f}\|_{L^2(I)}^2 + \sum_{\ell=0,1} \|\hat{G}^\ell\|_{H^{2-\ell}(I)}^2 + \sum_{\ell=0,1} |\eta|^{3-2\ell}(|\hat{g}_0^\ell|^2 + |\hat{g}_\omega^\ell|^2))$$

where C_2 and C_3 are some constants independent of \hat{u} and η . Thus we have proved (4.17) for η on the whole line $\text{Im } \eta = b$ on which $\mathcal{R}(\eta)$ has no pole. \square

Lemma 4.4. Let \mathcal{T} be a strip $= \{\eta | -h < \text{Im } \eta < h\}$, $0 < h < \kappa_1$, then $\mathcal{R}(\eta)$ has no pole in \mathcal{T} for the Dirichlet and mixed problems, and the origin is the only pole of $\mathcal{R}(\eta)$ in \mathcal{T} for the Neumann problem.

Proof. Due to the definition of κ_1 and Theorem 6.1 of [1] $\mathcal{R}(\eta)$ may have poles on a finite segment of the real axis. We shall show that there is no pole of $\mathcal{R}(\eta)$ on the real axis for the Dirichlet and mixed problems and the origin is the only pole of $\mathcal{R}(\eta)$ on the real axis for the Neumann problem.

(1) Suppose that $\eta = \xi$ (real) is the pole of $\mathcal{R}(\eta)$ for the Dirichlet problem, ξ satisfies

$$\text{sh} \xi \omega = \pm \frac{\sin \omega}{3-4\nu} \xi.$$

But Lemma A.1 shows that for $\xi = 0$ and $\omega = k\pi$, there is no non-trivial solution of (4.8) and (4.9a). Hence we may assume that $\omega \neq k\pi$. Let $f(\xi) = \text{sh} \omega \xi - |\sin \omega| \xi$, then $f'(\xi) = \omega \text{ch} \xi \omega - |\sin \omega| \geq \omega - |\sin \omega| > 0$. Hence $f(\xi) > 0$ for $0 < \xi < \infty$. Similarly, we can show that $f'(\xi) < 0$ for $-\infty < \xi < 0$. Therefore $|\text{sh} \xi \omega| > |\sin \omega| |\xi| > \frac{1}{3-4\nu} |\sin \omega| |\xi|$ for real $\xi \neq 0$ and $\nu \in (0, 1/2)$, which implies that the zero is the only real number which satisfies (4.12). Since zero is not an eigenvalue, $\mathcal{R}(\eta)$ has no pole on the real line for the problem with the Dirichlet boundary condition.

The proof for the mixed boundary condition is similar.

(ii) Suppose $\eta = \xi$ (real) is a pole of $\mathcal{R}(\eta)$ for the Neumann problem. Then ξ satisfies

$$\operatorname{sh} \xi \omega = \pm \xi \sin \omega.$$

It has been proven in (i) that $|\operatorname{sh} \xi \omega| > |\xi| |\sin \omega|$ for $\xi \in (-\infty, \infty)$ and $\xi \neq 0$, hence, on the real line, (4.13) is satisfied only at the origin. Lemma A.2 has shown that $\xi = 0$ is the eigenvalue with multiplicity of 2, the corresponding characteristic subspace is spanned by $\bar{e}_1 = (\cos \theta, -\sin \theta)^T$ and $\bar{e}_2 = (\sin \theta, \cos \theta)^T$. \square

If $h \neq 0$ is selected in $(-\kappa_1, \kappa_1)$, there is no pole of $\mathcal{R}(\eta)$ for the Dirichlet, Neumann and mixed problems on the line $\operatorname{Im} \eta = h$. As a consequence of Theorem 4.1 and Lemma 4.4 we have Theorem 4.2 and Theorem 4.3.

Theorem 4.2. If $\tilde{f} \in H_h^0(D)$, $\tilde{G}^\ell \in H_h^{2-\ell}(D)$, $\ell = 0, 1$, $0 < h < \kappa_1$, then equation (4.4) with boundary conditions (4.5) has a unique solution $\tilde{u} \in H_h^2(D)$ and $|\alpha| \leq 2$

$$(4.18) \quad \|D^{\alpha} \tilde{u}\|_{L_h(D)} \leq C \left[\|\tilde{f}\|_{H_h^0(D)} + \sum_{\ell=0,1} \|\tilde{G}^\ell\|_{H_h^{2-\ell}(D)} \right]$$

$\|\tilde{G}^0\|_{H_h^2(D)}$ (resp. $\|\tilde{G}^1\|_{H_h^1(D)}$) is absent in (4.18) for the Neumann problem

(resp. the Dirichlet problem).

Theorem 4.3. If $\bar{f} \in L_\beta(Q)$, $\bar{G}^1 \in W_\beta^{2-1}(Q)$, $1 = 0, 1$, $0 < \beta < 1$, $\beta > 1 - \kappa_1$, and then the equation (4.1) with boundary condition (4.3) has the unique solution \bar{u} in $W_\beta^2(Q)$, and

$$(4.19) \quad \|\bar{u}\|_{W_\beta^2(Q)} \leq C \left[\|\bar{f}\|_{L_\beta(Q)} + \sum_{\ell=0,1} \|\bar{G}^\ell\|_{W_\beta^{2-\ell}(Q)} \right]$$

where $\|\bar{G}^0\|_{W_{\beta}^2(Q)}$ (resp. $\|\bar{G}^1\|_{W_{\beta}^1(Q)}$) drops out in (4.19) for the Neumann problem (resp. the Dirichlet problem).

Remark 4.1. The proof of Theorem 4.2 and 4.3 is similar to those for the Poisson equation in [2]. The shift theorem in the space $W_{\beta,p}^k(Q)$ for general elliptic systems was given in [21], where the Agranovich-Vishik's conditions are used, and the shift theorem in $W_{\beta,p}^k(Q)$ for the plane elasticity problem was given in [26] without verifying the conditions.

Remark 4.2. The shift theorem can be easily generalized to any $k > 2$, namely

$$\|\bar{u}\|_{W_{\beta}^k(Q)} \leq C \left[\|\bar{f}\|_{W_{\beta}^{k-2}(Q)} + \sum_{\ell=0,1} \|\bar{G}^{\ell}\|_{W_{\beta}^{k+2-\ell}(Q)} \right]$$

provided $\bar{f} \in W_{\beta}^{K-2}(Q)$, $\bar{G}^{\ell} \in W_{\beta}^{k-\ell}(Q)$ and there is no pole of $\mathcal{R}(\eta)$ on the line $\text{Im } \eta = k + 1 - \beta$. But in practical problems the shift theorem is less applicable for $k > 2$ because of very strong conditions on \bar{f} and \bar{G}^{ℓ} in the neighborhood of the origin, for instance when \bar{f} and \bar{G}^{ℓ} are analytic and vanish rapidly at ∞ , but do not belong to $W_{\beta}^k(Q)$, $k \geq 2$. Hence the shift theorem for $k > 2$ is not directly applicable to these problems with analytic or piecewise analytic data (see [22]). For this reason we addressed the shift theorem in the space $W_{\beta}^k(Q)$ only for $k = 2$, and will address it in the space $H_{\beta}^{k,\ell}(Q)$ for $k > 2$ in Section 5.

Corollary 4.1. If \bar{G}^0 vanishes at the origin and $\bar{G}^{\ell} \in H_{\beta}^{2-\ell,2-\ell}(Q)$ then (4.19) can be rewritten as

$$(4.19') \quad \|\bar{u}\|_{W_{\beta}^2(Q)} \leq C \left[\|\bar{f}\|_{L_{\beta}(Q)} + \sum_{\ell=0,1} \|\bar{G}^{\ell}\|_{H_{\beta}^{2-\ell,2-\ell}(Q)} \right].$$

Proof. By Lemma A, 2 of [2] $\bar{G}^{\ell} \in W_{\beta}^{2-\ell}(Q)$, $\ell = 0, 1$, and

$$\|\bar{G}\|_{W_{\beta}^{2-l}(Q)}^{\ell} \leq C \|\bar{G}\|_{H^{2-l,2-l}(Q)}.$$

Then (4.19') follows easily from (4.19). \square

Corollary 4.2. Let $u = A^{-1}\bar{u}$ denote the displacement in Cartesian coordinates and $\bar{u} \in W_{\beta}^2(Q)$, $0 < \beta < 1$ be the solution of (4.1) and (4.3). Then $u \in W_{\beta}^2(Q)$, and there are constants c_1 and $c_2 > 0$ independent of u and \bar{u} such that

$$(4.20) \quad c_1 \|\bar{u}\|_{W_{\beta}^2(Q)} < \|u\|_{W_{\beta}^2(Q)} < c_2 \|\bar{u}\|_{W_{\beta}^2(Q)}.$$

Proof. Since

$$\begin{aligned} u_1 &= u_r \cos \theta - u_{\theta} \sin \theta \\ u_2 &= u_r \sin \theta + u_{\theta} \cos \theta \end{aligned}$$

obviously $\|r^{\beta-2}u\|_{L^2(Q)} = \|r^{\beta-2}\bar{u}\|_{L^2(Q)} < \|\bar{u}\|_{W_{\beta}^2(Q)}$. Further note that

$$\begin{aligned} \frac{1}{r^2} \frac{\partial^2 u_1^2}{\partial \theta^2} &= \frac{1}{r^2} \left(\frac{\partial^2 u_r}{\partial \theta^2} \cos \theta - \frac{\partial^2 u_{\theta}}{\partial \theta^2} \sin \theta \right) - \frac{2}{r^2} \left(\frac{\partial u_r}{\partial \theta} \sin \theta - \frac{\partial u_{\theta}}{\partial \theta} \cos \theta \right) \\ &\quad - \frac{1}{r^2} (u_r \cos \theta + u_{\theta} \sin \theta). \end{aligned}$$

Then we have $\|r^{\beta-2} \frac{\partial^2 u_1}{\partial \theta^2}\|_{L^2(Q)} \leq C \|\bar{u}\|_{W_{\beta}^2(Q)}^2$. Similarly we can prove that for

$0 \leq k \leq 2$,

$$\|r^{\beta-2+k} \partial^k u_1\|_{L_{\beta}^2(Q)}^2 \leq C \|\bar{u}\|_{W_{\beta}^2(Q)}^2$$

which yields

$$\|u\|_{W_{\beta}^2(Q)} \leq C_2 \|\bar{u}\|_{W_{\beta}^2(Q)}.$$

In the exactly same way we can prove the other half of (4.20). \square

For the regularity of the weak solution of (3.1) we are interested in the special auxiliary problem for which f and G^ℓ vanish for $r > 1$.

Theorem 4.4. Let $\bar{u} \in W_\beta^2(Q)$ be the solution of (4.1) and (4.3) with $\bar{f} \in L_\beta(Q)$, $\bar{G}^\ell \in H_\beta^{2-\ell, 2-\ell}(Q)$, $\ell = 0, 1$, $0 < \beta < 1$, $\beta > 1 - \kappa_1$, and let $u = A^{-1}\bar{u}$ be the displacement in Cartesian Coordinates. If \bar{f} and \bar{G}^ℓ , $\ell = 0, 1$ vanish for $r > 1$, and \bar{G}^0 vanishes for $r = 0$, then for the problem with the Dirichlet boundary condition (4.3a) and mixed boundary condition (4.3c)

$$(4.21a) \quad \|\mathcal{D}^1 \bar{u}\|_{L^2(Q)}^2 + \|r^{-1} \bar{u}\|_{L^2(Q)}^2 < \infty,$$

$$(4.21b) \quad \|\mathcal{D}^1 u\|_{L^2(Q)}^2 + \|r^{-1} u\|_{L^2(Q)}^2 < \infty,$$

and for the problem with the Neumann boundary condition (4.3b)

$$(4.22) \quad \|\mathcal{D}^1 u\|_{L^2(Q)}^2 < \infty.$$

Proof. We first prove (4.21) for the Dirichlet problem. Select β' such that $1 < \beta' < 1 + \kappa_1$. Then $\beta' > 1 > \beta$ and $-\kappa_1 < 1 - \beta' < 0 < 1 - \beta < \kappa_1$. Due to Lemma 4.4 $\mathcal{R}(\eta)$ has no pole between the line $\text{Im } \eta = 1 - \beta$ and the line $\text{Im } \eta = 1 - \beta'$. Because \bar{f} and \bar{G}^ℓ , $\ell = 0, 1$ vanish for $r > 1$ and $\bar{G}_0 = 0$ at the origin, $\bar{f} \in L_{\beta'}(Q)$ and $\bar{G}^\ell \in W_{\beta'}^{2-\ell}(Q)$, $\ell = 0, 1$. By the argument of Theorem 4.3, $\bar{u} \in W_{\beta'}^2(Q)$. Thus we have

$$(4.23) \quad \int_0^\omega \int_1^\infty \left[|\mathcal{D}^1 \bar{u}|^2 + |r^{-1} \bar{u}|^2 \right] r dr d\theta \\ \leq \int_0^\omega \int_1^\infty \left[r^{2(\beta'-1)} |\mathcal{D}^1 \bar{u}|^2 + r^{2(\beta'-2)} |\bar{u}|^2 \right] r dr d\theta \leq \|\bar{u}\|_{W_{\beta'}^2(Q)}^2.$$

On other hand we have

$$\begin{aligned}
(4.24) \quad & \int_0^\omega \int_0^1 \left[|\mathcal{D}^1 \bar{u}|^2 + |r^{-1} \bar{u}|^2 \right] r dr d\theta \\
& \leq \int_0^\omega \int_0^1 \left[r^{2(\beta-1)} |\mathcal{D}^1 \bar{u}|^2 + r^{2(\beta-2)} |\bar{u}|^2 \right] r dr d\theta \leq \|\bar{u}\|_{W_{\beta'}^2(Q)}^2
\end{aligned}$$

which together with (4.23) yields (4.21a). Noting that

$$\begin{aligned}
|\mathcal{D}^1 u|^2 &= |\mathcal{D}^1 \bar{u}|^2 + |r^{-1} \bar{u}|^2 + \frac{2}{r^2} \left[\frac{\partial u_\theta}{\partial \theta} u_r - \frac{\partial u_r}{\partial \theta} u_\theta \right] \leq 2 \left[|\mathcal{D}^1 \bar{u}|^2 + |r^{-1} \bar{u}|^2 \right] \\
|r^{-1} u|^2 &= |r^{-1} \bar{u}|^2
\end{aligned}$$

we have (4.21b) immediately.

The proof for the mixed problem is similar and will not again be elaborated here. Next we prove (4.22) for the Neumann problem. Let β' be selected as above, then by Lemma 4.4 the origin is the only pole of $\mathcal{R}(\eta)$ between the line $\text{Im } \eta = 1 - \beta$ and the line $\text{Im } \eta = 1 - \beta'$. Let $\bar{u}^* = \tilde{u}^*(\ln \frac{1}{r}, \theta)$ and

$$\tilde{u}^*(\tau, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty + i(1-\beta')}^{\infty + i(1-\beta')} \mathcal{R}(\eta) [\hat{f}, \hat{g}^1] e^{i\eta\tau} d\eta.$$

Arguing as above \bar{u}^* is a solution of the Dirichlet problem (4.1) and (4.3b) in $W_{\beta'}^2(Q)$. By Theorem 2 of [26]

$$\bar{u} = \bar{u}^* + a_1 \bar{e}_1 + a_2 \bar{e}_2$$

where \bar{e}_1 and \bar{e}_2 are defined as in Lemma 4.4. Let $u = A^{-1} \bar{u}$ and $u^* = A^{-1} \bar{u}^*$, then

$$u = u^* + (c_1, c_2)^T$$

and

$$|\mathcal{D}^1 u|^2 = |\mathcal{D}^1 u^*|^2$$

Note that

$$\begin{aligned}
 (4.25) \quad \int_0^\omega \int_0^1 |\mathcal{D}^1 u|^2 r dr d\theta &= \int_0^\omega \int_0^1 \left(|\mathcal{D}^1 \bar{u}|^2 + |r^{-1} \bar{u}|^2 \right) r dr d\theta \\
 &\leq \int_0^\omega \int_0^1 \left(r^{2(\beta-1)} |\mathcal{D}^1 \bar{u}|^2 + r^{2(\beta-2)} |\bar{u}|^2 \right) r dr d\theta \\
 &\leq \|\bar{u}\|_{W_\beta^2(Q)}^2
 \end{aligned}$$

and

$$\begin{aligned}
 (4.26) \quad \int_0^\omega \int_1^\infty |\mathcal{D}^1 u|^2 r dr d\theta &= \int_0^\omega \int_1^\infty |\mathcal{D}^1 u^*|^2 r dr d\theta \\
 &\leq \int_0^\omega \int_1^\infty \left(r^{2(\beta'-1)} |\mathcal{D}^1 \bar{u}^*|^2 + r^{2(\beta'-2)} |\bar{u}^*|^2 \right) r dr d\theta \\
 &\leq \|\bar{u}^*\|_{W_{\beta'}^2(Q)}^2.
 \end{aligned}$$

Combining (4.25) and (4.26) we get (4.22). \square

Since $|\mathcal{D}^1 u|^2 = |D^1 u|^2$, $|\mathcal{D}^1 \bar{u}|^2 = |D^1 \bar{u}|^2$ we have the following corollary.

Corollary 4.3. Let u and \bar{u} be the same as in Theorem 4.4. Then for the problem with Dirichlet condition (4.3a) and mixed condition (4.3c)

$$(4.27a) \quad \|\mathcal{D}^1 \bar{u}\|_{L^2(Q)}^2 + \|r^{-1} \bar{u}\|_{L^2(Q)}^2 < \infty,$$

$$(4.27b) \quad \|D^1 u\|_{L^2(Q)}^2 + \|r^{-1} u\|_{L^2(Q)}^2 < \infty,$$

and for the problems with Neumann condition (4.3b)

$$(4.28) \quad \|D^1 u\|_{L^2(Q)} < \infty. \quad \square$$

Remark 4.3. (4.21) and (4.27) may not hold for the problem with the Neumann condition, (4.22) and (4.28) may not hold for \bar{u} .

§5. REGULARITY OF SOLUTION OF THE ELASTICITY PROBLEM

We shall first discuss the regularity of the weak solution of the problem (3.1) over the polygonal domain Ω with $f \in L_\beta(\Omega)$ and $G^\ell \in H_\beta^{2-\ell, 2-\ell}(\Omega)$, namely the relationship between the weak solution and the solution of the auxiliary problem over an infinite sector Q with f and G^ℓ having a bounded support. Then we shall derive the regularity of the solution of the problem (3.1) in the countable weighted Sobolev space.

Assume that A_1 is located at the origin and Γ_1 lies in positive x_1 -axis. Let (r, θ) denote polar coordinates, and let $\delta \in (0, 1)$ and $S_\delta = \{(r, \theta) | 0 < r < \delta, 0 < \theta < \omega\} \subset \Omega$. Let $\phi_\delta(r)$ be a cut-off function in $C^\infty(R^1)$ such that $\phi_\delta \equiv 1$ for $0 < r < \frac{\delta}{2}$ and $\phi_\delta \equiv 0$ for $r > \delta$. If $\Gamma_1 \cup \Gamma_M \subset \Gamma_1$, let $v = (v_1, v_2)^T = \phi_\delta(r)u$, otherwise $v = \psi_\delta(r)(u - G^0(A_1))$ where $u = (u_1, u_2)^T \in H^1(\Omega)$ is the weak solution of (3.1). v is extended to $Q = S_\infty = \{(r, \theta) | 0 < r < \infty, 0 < \theta < \omega\}$ by zero extension outside S_δ .

Theorem 5.1. Let u be the weak solution of the problem (3.1) with $f \in L_\beta(\Omega)$ and $G^\ell \in H_\beta^{2-\ell, 2-\ell}(\Omega)$, $0 < \beta > 1$, $\beta > 1 - \kappa_1$, and let $v = \phi_\delta u$ if $\Gamma_1 \cup \Gamma_M \subset \Gamma^1$ and $v = \phi_\delta(u - G^0(A_1))$ otherwise. Then

(i) If $\Gamma_1 \subset \Gamma^0$ or $\Gamma_M \subset \Gamma^0$, or $\Gamma_1 \cup \Gamma_M \subset \Gamma^0$, then $v \in W_\beta^2(Q)$, and

$$(5.1a) \quad \|v\|_{W_\beta^2(Q)} \leq C \left[\|f\|_{L_\beta(S_\delta)} + \sum_{\ell=1}^2 \|G^\ell\|_{H_\beta^{2-\ell, 2-\ell}(S_\delta)} + \|u\|_{H^1(S_\delta \setminus S_{\delta/2})} \right].$$

(ii) If $\Gamma_1 \cup \Gamma_M \subset \Gamma^1$, then $(v - \sum_{i=1,2} c_i e_i) \in W_\beta^2(Q)$, where $e_1 =$

$(1, 0)^T$, $e_2 = (0, 1)^T$, c_i , $i = 1, 2$ are some constants, and

$$(5.1b) \quad \|v - \sum_{i=1,2} c_i e_i\|_{W_\beta^2(Q)} \leq C \left[\|f\|_{L_\beta(S_\delta)} + \|G^1\|_{H_\beta^{1,1}(S_\delta)} + \|u\|_{H^1(S_\delta \setminus S_{\delta/2})} \right].$$

Proof. (i) We prove (5.1a) for the case $\Gamma_1 \cup \Gamma_M \subset \Gamma^0$ (Dirichlet boundary type), the proof for the mixed boundary type is similar with what follows.

At first we may assume that $G^0 = 0$ at A_1 , then v satisfies

$$(5.2) \quad \begin{cases} L(v) = \phi_\delta f + L_1(u, \phi_\delta) = \tilde{f} & \text{in } Q \\ v|_{\theta=0, \omega} = \phi_\delta G^0|_{\theta=0, \omega} = \tilde{G}^0|_{\theta=0, \omega} \end{cases}$$

where $L_1(u, \phi_\delta)$ is a sum of terms $D^\alpha u D^{\alpha'} \phi_\delta$, with $0 \leq |\alpha| \leq 1$, $1 \leq |\alpha'| \leq 2$. Obviously \tilde{f} , \tilde{G}^0 and v vanish for $r > \delta$ and

$$(5.3) \quad \|\tilde{f}\|_{L_\beta(Q)} \leq C \left[\|\tilde{f}\|_{L_\beta(S_\delta)} + \|u\|_{H^1(S_\delta \setminus S_{\delta/2})} \right].$$

For $w \in H_0^1(Q) = \{w \mid \|D^1 w\|_{L^2(Q)} < \infty, w|_{\theta=0, \omega} = 0\}$ we have

$$B(v, w) = \int_Q \tilde{f} \cdot w \, r dr d\sigma, \quad \forall w \in H_0^1(Q)$$

where B is the bilinear form defined in (3.4).

On the other hand, by Theorem 4.3 and Corollary 4.1~4.2 there exists a unique solution $z = (z_1, z_2)^T$ of (5.2) in $W_\beta^2(Q)$, and

$$(5.4) \quad \begin{aligned} \|z\|_{W_\beta^2(Q)} &\leq C \left[\|\tilde{f}\|_{L_\beta(Q)} + \|\tilde{G}^0\|_{H_\beta^2(Q)} \right] \\ &\leq C \left[\|f\|_{L_\beta(S_\delta)} + \|G^0\|_{H_\beta^{2,2}(S_\delta)} + \|u\|_{H^1(S_\delta \setminus S_{\delta/2})} \right], \end{aligned}$$

and owing to Corollary 4.3 $\|D^1 z\|_{L^2(Q)} < \infty$.

Let $\tilde{H}_0^1(Q) = \{w \mid w \in H_0^1(Q) \text{ with bounded support in } Q\}$. Thus we have

$$B(z, w) = \int_Q \tilde{f} \cdot w \, r dr d\theta, \quad \forall w \in \tilde{H}_0^1(Q);$$

hence

$$B(z-v, w) = 0 \quad \forall w \in \tilde{H}_0^1(Q).$$

Since $\tilde{H}_0^1(Q)$ is dense in $H_0^1(Q)$, $B(z-v, z-v) = 0$. Therefore $(z-v)$ can only be a rigid body motion. Due to the Dirichlet boundary condition

$$v = z \in W_\beta^2(Q).$$

If $G^0 \neq 0$ at the origin, let $\tilde{u} = u - G^0(A_1)$ and $v = \phi_\delta \tilde{u}$. Then we have

$$L(v) = \phi_\delta f + L_1(u, \phi_\delta) = \tilde{f} \quad \text{in } Q$$

$$v|_{\theta=0, \omega} = \phi_\delta (G^0(r, \theta) - G^0(A_1)) \Big|_{\theta=0, \omega} = \tilde{G} \Big|_{\theta=0, \omega}.$$

Obviously $\tilde{f} \in L_\beta(Q)$, \tilde{f} and \tilde{G}^0 vanishes in Q for $r > \delta$. Applying the results above to v we have that $v = \phi_\delta(u - G^0(A_1)) \in W_\beta^2(Q)$ and (5.1a) holds.

(ii) If $\Gamma_1 \cup \Gamma_M \subset \Gamma^1$, v satisfies

$$L(v) = \phi_\delta f + L_1(u, \phi_\delta) = \tilde{f} \quad \text{in } Q$$

(5.5)

$$T(v) \Big|_{\theta=0, \omega} = \phi_\delta g^1 + \ell_1(u, \phi_\delta) \Big|_{\theta=0, \omega} = \tilde{g}^1 = \tilde{G}^1 \Big|_{\theta=0, \omega}$$

where $L_1(u, \phi_\delta)$, \tilde{f} are the same as in (i), and $\ell_1(u, \phi_\delta)$ consists of terms $u D^\alpha \phi_\delta$, $|\alpha| = 1$. Obviously $\tilde{G}^1 \in W_\beta^1(Q)$, and

$$(5.6) \quad \|\tilde{G}^1\|_{W_\beta^1(Q)} \leq C \left[\|G^1\|_{H_\beta^{1,1}(S_\delta)} + \|u\|_{H^1(S_\delta \setminus S_{\delta/2})} \right],$$

and \tilde{f}, \tilde{G}^1 and v vanish for $r > \delta$. For any $w \in \tilde{H}^1(Q) = \{w \mid \|D^1 w\|_{L^2(Q)} < \infty\}$

$$B(v, w) = \int_Q \tilde{f} \cdot w \, dx + \int_{\partial Q} \tilde{g}^1 \cdot w \, ds.$$

On the other hand, by Theorem 4.3 and Corollary 4.1~4.2 there is a unique solution $z = (z_1, z_2)^T$ of (5.5) in $W_\beta^2(Q)$ and

$$(5.7) \quad \|z\|_{W_{\beta}^2(Q)} \leq C \left[\|\tilde{f}\|_{L_{\beta}(Q)} + \|\tilde{G}^1\|_{W_{\beta}^1(Q)} \right]$$

by (5.3) and (5.6)

$$\leq C \left[\|f\|_{L_{\beta}(S_{\delta})} + \|G^1\|_{H_{\beta}^{1,1}(Q)} + \|u\|_{H^1(S_{\delta} \setminus S_{\delta/2})} \right].$$

and by Theorem 4.4 $\|D^1 z\|_{L^2(Q)} < \infty$.

Let $\tilde{H}^1(Q) = \{w | w \in \tilde{H}^1(Q) \text{ with bounded support in } Q\}$, then z satisfies

$$B(z, w) = \int_Q \tilde{f} w dx + \int_{\partial Q} \tilde{g}^1 \cdot w ds, \quad \forall w \in \tilde{H}^1(Q).$$

Therefore

$$B(z - v, w) = 0, \quad \forall w \in \tilde{H}^1(Q).$$

Since $\tilde{H}^1(Q)$ is dense in $\tilde{H}^1(Q)$ we have

$$B(z - v, z - v) = 0,$$

which indicates that the strain energy is zero, $z - v$ represents only a rigid body motion, i.e.,

$$z - v = \sum_{i=1}^3 c_i e_i$$

where $e_3 = (-y, x)^T$, and c_i , $i = 1, 2, 3$ are constants. Because $D^1(z - v) \in L^2(Q)$, $c_3 = 0$. Then (5.1b) follows from (5.7).

Corollary 5.1. v is continuous in \bar{Q} , and $(v - v(A_1)) \in W_{\beta}^2(Q)$.

Proof. If $\Gamma_1 \cup \Gamma_M \subset \Gamma^1$, by Theorem 5.1, $(v - \sum_{i=1}^2 c_i e_i) \in H_{\beta}^{2,2}(S_{R_0})$ for any $R_0 > 0$. Hence $(v - \sum_{i=1}^2 c_i e_i) \in C^0(\bar{S}_{R_0})$ (see [2]), therefore v is continuous in \bar{Q} . Since $(v - \sum_{i=1}^2 c_i e_i) \in W_{\beta}^2(Q)$, $0 < \beta < 1$, $r^{\beta-2}(v - \sum_{i=1}^2 c_i e_i) \in L^2(\bar{S}_{R_0})$

which implies that $\sum_{i=1}^2 c_i e_i = v(A_1)$ and that $(v - v(A_1)) \in W_\beta^2(Q)$.

If $\Gamma_1 \subset \Gamma^0$ or $\Gamma_M \subset \Gamma^0$ by Theorem 5.1 $v = \phi_\delta(u - G^0) \in W_\beta^2(Q)$. Hence $v \in H_\beta^{2,2}(S_{R_0})$ for any $R_0 > 0$, thus $v - G^0(A_1) \in C^0(\bar{S}_{R_0})$, and v is continuous in \bar{Q} and $v(A_1) = G^0(A_1)$, and $(v - v(A_1)) \in W_\beta^2(Q)$.

Corollary 5.2. Let $u = (u_1, u_2)^T$ and $\bar{u} = (u_r, u_\theta)^T$ be the weak solution of (4.1) in Cartesian and polar coordinates, respectively. Then $(u - u(A_1)) \in W_\beta^2(S_{\delta/2})$ and $(\bar{u} - \bar{u}(A_1)) \in W_\beta^2(S_{\delta/2})$, and

$$(5.8a) \|u - u(A_1)\|_{W_\beta^2(S_{\delta/2})} \leq C \left[\|f\|_{L_\beta(S_\delta)} + \sum_{\ell=0}^1 \|G^\ell\|_{H_\beta^{2-\ell, 2-\ell}(S_\delta)} + \|u\|_{H^1(S_\delta \setminus S_{\delta/2})} \right],$$

$$(5.8b) \|\bar{u} - \bar{u}(A_1)\|_{W_\beta^2(S_{\delta/2})} \leq C \left[\|\bar{f}\|_{L_\beta(S_\delta)} + \sum_{\ell=0}^1 \|\bar{G}^\ell\|_{H_\beta^{2-\ell, 2-\ell}(S_\delta)} + \|\bar{u}\|_{H^1(S_\delta \setminus S_{\delta/2})} \right].$$

Proof. Note that $v = u$ and $\bar{v} = \bar{u}$ in $\bar{S}_{\delta/2}$, then (5.8a) and (5.8b) follow easily from Theorem 5.1 and Corollary 5.1.

Remark 5.1. The regularity of the weak solution of the elasticity problem in polynomial domain was addressed in [26]. It was concluded in [26] that $v =$

$\phi_\delta u \in W_\beta^2(Q)$ with $\beta = 1 + \varepsilon$, $\varepsilon > 0$ arbitrary provided $f \in L_\beta(Q)$,

$G^\ell \in W^{2-\ell, 2-\ell}(Q)$, and $\mathcal{R}(\eta)$ has no pole on the line $\text{Im } \eta = -\varepsilon$. Actually the

condition $\beta > 1$ is not necessary. Theorem 5.1 indicates that $v \in H_\beta^{2,2}(Q)$

or $(v - v(A_1)) \in W_\beta^2(Q)$ if $\beta > 1 - \kappa_1$. κ_1 is the smallest positive

imaginary part of eigenvalues of the operator $\mathcal{R}(\eta)$, which depends on the

geometry of the domain, the type of boundary conditions and the material

properties. The condition $\beta > 1 - \kappa_1$ precisely reflects the nature of the

singularity of v . If $\kappa_1 \leq 1$, then $v \in H_{\beta}^{2,2}(Q)$ with $\beta \in (1 - \kappa_1, 1)$, or if $\kappa_1 > 1$ then $v \in H^2(Q)$, and v can be even smoother. We will elaborate on this later. Nevertheless there always exists some $\beta \in (0, 1)$ such that $v \in H_{\beta}^{2,2}(Q)$ and $(v - v(A_1)) \in W_{\beta}^2(Q)$. The improvement above is substantial.

Lemma 5.1. For $\beta \in (0, 1)$, $k \geq \ell \geq 0$, $0 \leq \ell < 2$ and $\bar{u}(A_1) = 0$ there exists $c_1, c_2 > 0$ such that

$$(5.9) \quad c_1 \|\bar{u}\|_{\mathcal{H}_{\beta}^{k,\ell}(S_{\delta})} \leq \|u\|_{\mathcal{H}_{\beta}^{k,\ell}(S_{\delta})} \leq c_2 \|\bar{u}\|_{\mathcal{H}_{\beta}^{k,\ell}(S_{\delta})}.$$

Moreover, if for some constant $\bar{C}, \bar{d} \geq 1$ and $\ell \leq |\bar{\alpha}| \leq k$

$$(5.10a) \quad \|r^{\bar{\alpha}_1 - \ell + \beta} D^{\bar{\alpha}} \bar{u}\|_{L^2(S_{\delta})} \leq \bar{C} \bar{d}^{k-\ell} (k-\ell)!$$

then for $|\alpha| = k$ and $C \leq M\bar{C}$, $d \leq N\bar{d}$,

$$(5.10b) \quad \|r^{\alpha_1 - \ell + \beta} D^{\alpha} u\|_{L^2(S_{\delta})} \leq Cd^{k-\ell} (k-\ell)!$$

Vice versa, if (5.10b) holds for $\ell \leq |\alpha| \leq k$, then (5.10a) stands for $|\bar{\alpha}| = k$ with $\bar{C} \leq MC$ and $\bar{d} \leq Nd$.

Proof. Note that

$$\begin{aligned} \frac{\partial^k u_1}{\partial r^k} &= \cos \theta \frac{\partial^k u_r}{\partial r^k} - \sin \theta \frac{\partial^k u_{\theta}}{\partial r^k} \\ \frac{\partial^k u_2}{\partial r^k} &= \sin \theta \frac{\partial^k u_r}{\partial r^k} + \cos \theta \frac{\partial^k u_{\theta}}{\partial r^k}. \end{aligned}$$

Then

$$\begin{aligned} (5.11) \quad \|r^{k-\ell+\beta} \frac{\partial^k u}{\partial r^k}\|_{L^2(S_{\delta})} &= \|r^{k-\ell+\beta} \frac{\partial^k \bar{u}}{\partial r^k}\|_{L^2(S_{\delta})} \\ &\leq \|\bar{u}\|_{\mathcal{H}_{\beta}^{k,\ell}(S_{\delta})} \end{aligned}$$

which is the second inequality of (5.9) for $\alpha_1 = |\alpha| = k$. For $\alpha_2 = |\alpha| = k$

we have

$$\begin{aligned}\frac{\partial^k u_1}{\partial \theta^k} &= \sum_{m=0}^k \binom{k}{m} \left[\frac{d^{k-m} \cos \theta}{d\theta^{k-m}} \frac{\partial^m u_r}{\partial \theta^m} - \frac{d^{k-m} \sin \theta}{d\theta^{k-m}} \frac{\partial^m u_\theta}{\partial \theta^m} \right], \\ \frac{\partial^k u_2}{\partial \theta^k} &= \sum_{m=0}^k \binom{k}{m} \left[\frac{d^{k-m} \sin \theta}{d\theta^{k-m}} \frac{\partial^m u_r}{\partial \theta^m} + \frac{d^{k-m} \cos \theta}{d\theta^{k-m}} \frac{\partial^m u_\theta}{\partial \theta^m} \right].\end{aligned}$$

Therefore

$$(5.12) \quad \|r^{\beta-\ell} \frac{\partial^k u}{\partial \theta^k}\|_{L^2(S_\delta)} \leq 2 \sum_{m=0}^k \binom{k}{m} \|r^{\beta-\ell} \frac{\partial^m \bar{u}}{\partial \theta^m}\|_{L^2(S_\delta)}.$$

Since $0 < \beta < 1$ and $\bar{u}(A_1) = 0$, due to Lemma A.2 of [2] we have

$$\begin{aligned}\|r^{\beta-1} \bar{u}\|_{L^2(S_\delta)} &\leq C_0 \|\bar{u}\|_{\mathcal{H}_\beta^{1,1}(S_\delta)}, \\ \|r^{\beta-2} \bar{u}\|_{L^2(S_\delta)} &\leq C_0 \|\bar{u}\|_{\mathcal{H}_\beta^{1,1}(S_\delta)}, \\ \|r^{\beta-2} \frac{\partial \bar{u}}{\partial \theta}\|_{L^2(S_\delta)} &\leq C_0 \|\bar{u}\|_{\mathcal{H}_\beta^{2,2}(S_\delta)}.\end{aligned}$$

Hence for $0 \leq \ell \leq 2$, $k \geq \ell$, we have

$$\begin{aligned}(5.13) \quad \|r^{\beta-\ell} \frac{\partial^k u}{\partial \theta^k}\|_{L^2(S_\delta)} &\leq C_2 \sum_{m=\ell}^k \binom{k}{m} \|r^{\beta-\ell} \frac{\partial^m \bar{u}}{\partial \theta^m}\|_{L^2(S_\delta)} \\ &\leq C_2 \|\bar{u}\|_{\mathcal{H}_\beta^{k,\ell}(S_\delta)}.\end{aligned}$$

Actually we can make similar arguments for each term of type $r^{\alpha_1-\ell+\beta} u$, $0 \leq \alpha_1 \leq |\alpha| = k$. Hence we have the second inequality of (5.9). The first inequality of (5.9) can be proved in exactly the same way.

Next we shall prove (5.10b) if (5.10a) holds. (5.10a) and (5.11) ~ (5.13) lead to

$$(5.14a) \quad \|r^{k-\ell+\beta} \frac{\partial^k u}{\partial r^k}\|_{L^2(S_\delta)} \leq \bar{C} d^{k-\ell} (k-\ell)!$$

and

$$(5.14b) \quad \left\| r^{\beta-\ell} \frac{\partial^k u}{\partial r^k} \right\|_{L^2(S_\delta)} \leq C_2 \bar{C} \sum_{m=\ell}^k \binom{k}{m} (m-\ell)! \bar{d}^{(m-\ell)} \\ \leq C_2 \bar{C} \left(1 + \frac{1}{\bar{d}}\right)^\ell (1+\bar{d})^{k-\ell} (k-\ell)!$$

which are (5.10b) for $\alpha_2 = 0$ and $\alpha_2 = |\alpha| = k$, respectively. Similar estimates can be derived for general terms $r^{\alpha_1-\ell+\beta} \mathcal{D}^\alpha u$, $0 \leq \alpha_1 \leq |\alpha| = k$.

Hence (5.10b) holds with $d \leq M\bar{d}$ and $C \leq N\bar{C}$ where $M = 2$ and $N = C_2 \bar{C} \left(1 + \frac{1}{\bar{d}}\right)^\ell$. In exactly the same way we can prove that (5.10a) holds for $|\bar{\alpha}| = k$ with $\bar{d} \leq Md$ and $\bar{C} \leq NC$ if (5.10b) holds for $\ell = |\alpha| \leq k$. \square

Theorem 5.2. Let $f \in H_\beta^{k,0}(\Omega)$, $g^\ell = G^\ell|_{\Gamma^\ell}$, $G^\ell \in H^{k+2-\ell}(\Omega)$, $\ell = 0, 1$, $\beta = (\beta_1, \beta_2, \dots, \beta_M)$, $0 < \beta_1 < 1$, $\beta_1 > 1 - \kappa_1^1$ (κ_1^1 is defined in (4.15) with respect to the vertex A_1), $1 \leq i \leq M$, and $|\Gamma^0| \neq 0$. Then the problem of (3.1) has a unique solution $u \in H_\beta^{k+2,2}(\Omega)$, and

$$(5.15) \quad \|u\|_{H_\beta^{k+2,2}(\Omega)} \leq C \left[\|f\|_{H_\beta^{k,0}(\Omega)} + \sum_{\ell=0,1} \|G^\ell\|_{H_\beta^{k+2-\ell,2-\ell}(\Omega)} \right].$$

Furthermore, if $f \in B_\beta^0(\Omega, C_f, d_f)$, $G^\ell \in B_\beta^{2-\ell}(\Omega, C_{G^\ell}, d_{G^\ell})$, $\ell = 0, 1$ then $u \in B_\beta^2(\Omega, C_u, d_u)$ with C_u and d_u satisfying

$$(5.16) \quad C_u \leq M_c (C_f + C_{G^0} + C_{G^1}),$$

$$d_u \leq M_d \max(d_f, d_{G^0}, d_{G^1})$$

where M_c and M_d are some constants independent of f and G^ℓ .

Remark 5.2. If $|\Gamma^0| = 0$, the theorem holds provided f and g^ℓ satisfy usual condition:

$$\int_{\Omega} f_i dx + \int_{\Gamma} g_i^1 ds = 0, \quad i = 1, 2$$

$$\int_{\Omega} (x_2 f_1 - x_1 f_2) dx + \int_{\Gamma} (x_2 g_1 - x_1 g_2) ds = 0.$$

Then the solution uniquely exists (up to a rigid body motion) in $H_{\beta}^{k+2,2}(\Omega)$ and $B_{\beta}^2(\Omega)$, respectively.

Proof. For the sake of simplicity, let Ω be a straight line polygon shown in Fig. 2.1. By Theorem 3.1, the problem (3.1) has the unique solution $u = (u_1, u_2)^T \in H^1(\Omega)$.

Let $S_{i,\delta_i} = \{(r_i, \theta_i) | 0 < r_i < \delta_i, 0 < \theta_i < \omega_i\} \subset \Omega$, shown in Fig. 5.1, where r_i and θ_i are the polar coordinates with respect to the vertex A_i and edge Γ_i . Assume that $0 < \delta_i < 1$ such that

$$S_{i,2\delta_i} \cap S_{j,2\delta_j} = \emptyset \quad \text{for } i \neq j, \quad i, j = 1, 2, \dots, M.$$

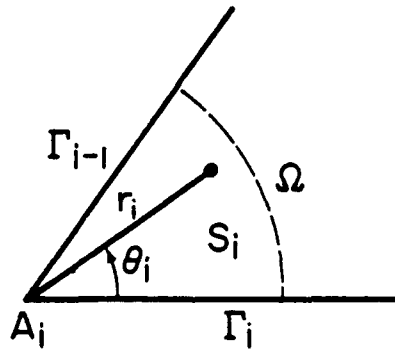


Fig. 5.1 Neighborhood of Vertex

Let $\Omega_{\delta} = \Omega \setminus \bigcup_{i=1}^M S_{i,\delta_i}$. By the argument of difference quotient it is easy to show that

$$(5.17) \quad \|u\|_{H^{k+2}(\Omega_{\delta/2})} \leq C \left[\|f\|_{H^k(\Omega_{\delta/4})} + \sum_{\ell=0,1} \|G^{\ell}\|_{H^{k+2-\ell}(\Omega_{\delta/4})} \right]$$

$$\leq C \left[\|f\|_{H_\beta^{k,0}(\Omega)} + \sum_{\ell=0,1} \|g^\ell\|_{H_\beta^{k+3/2-\ell, 3/2-\ell}(\Gamma^\ell)} \right].$$

If $f \in B_\beta^0(\Omega)$, $G^\ell \in B_\beta^{2-\ell}(\Omega)$, $\ell = 0, 1$, by Morrey's argument in [23] we can prove that u is analytic in Ω and on Γ except the vertices, and there exist some constants C_0 and d_0 satisfying

$$(5.18) \quad C_0 \leq M_0 (C_f + C_{G^0} + C_{G^1})$$

$$d_0 \leq N_0 \max(d_f, d_{G^0}, d_{G^1})$$

such that for $|\alpha| = k \geq 2$, $1 \leq i \leq M$

$$(5.19) \quad \|D^\alpha u\|_{L^2(\Omega_{\delta/2})} \leq C_0 d_0^{k-2} (k-2)!.$$

By the arguments of Theorem 2.1 and Lemma 5.1 we have

$$(5.20) \quad \|r_i^{-\alpha/2} \mathcal{D}^\alpha \bar{u}\|_{L^2(S_{i,\delta_i} \setminus S_{i,\delta_i/2})} \leq c_0 d_0^{k-2} (k-2)!$$

where C_0 and d_0 may be different from those in (5.19), but we use the same notations for simplicity.

By Theorem 2.1 and Lemma 5.1 $f, \bar{f} \in \mathcal{H}_\beta^{k,0}(\Omega)$ (resp. $\mathcal{B}_\beta^0(\Omega)$), $g^\ell, \bar{g}^\ell \in \mathcal{H}_\beta^{k+3/2-\ell, 3/2-\ell}(\Gamma^\ell)$ (resp. $\mathcal{B}^{3/2-\ell}(\Gamma^\ell)$), and it is sufficient to prove that in each sector $S_{i,\delta_i/2}$, $1 \leq i \leq M$

$$(5.21) \quad \|\bar{u}\|_{\mathcal{H}_\beta^{k+2,2}(S_{i,\delta_i/2})} \leq C_1 \left[\|\bar{f}\|_{\mathcal{H}_\beta^{k,0}(S_{\delta_i})} + \sum_{\ell=0,1} \|\bar{g}^\ell\|_{\mathcal{H}_\beta^{k+2,\ell,2-\ell}(S_{\delta_i})} + \|\bar{u}\|_{\mathcal{H}^{k+1}(S_{\delta_i} \setminus S_{\delta_i/2})} \right]$$

where C_1 depends on k , and

$$(5.22) \quad \| I_1^{\alpha_1-2} D_1^{\alpha_1-} \bar{u} \|_{L_{\beta_1}(S_{1,\delta_1/2})} \leq L_1 D_1^k P_1 (\overline{\alpha_2-2})_{k!}$$

respectively, where $|\alpha| = k + 2$, $k \geq 0$, L_1 , P_1 and D_1 are sufficiently large constants, but independent of k . $(\overline{\alpha_2-2}) = \alpha_2 - 2$ if $\alpha_2 \geq 2$ and $\overline{\alpha_2-2} = 0$ if $\alpha_2 < 2$.

By Corollary 5.2 we have

$$(5.23) \quad \|\bar{u} - \bar{u}(A_1)\|_{W_{\beta_1}^2(S_{1,\delta_1/2})} \leq C \left[\|\bar{f}\|_{L_{\beta_1}(S_{1,\delta_1})} + \sum_{\ell=0,1} \|\bar{G}^\ell\|_{H_{\beta_1}^{2-\ell,2-\ell}(S_{1,\delta_1})} + \|\bar{u}\|_{H^1(S_{1,\delta_1} \setminus S_{1,\delta_1/2})} \right].$$

We may assume that $i = 1$, A_1 is at the origin, Γ_1 lies on the positive x_1 -direction, and assume without losing generality that $\bar{u}(A_1) = 0$. To simplify the notation we shall write $S_{1,\delta_1} = S_\delta$, $\beta_1 = \beta$, etc. There are three cases to be considered:

$$(i) \quad \Gamma_1, \Gamma_M < \Gamma^0,$$

$$(ii) \quad \Gamma_1, \Gamma_M < \Gamma^1,$$

$$(iii) \quad \Gamma_1 < \Gamma^0, \Gamma_M < \Gamma^1.$$

In case (i) we assume $\bar{G}^0 = 0$, and let $\bar{v}^k = r^{k\frac{\partial^k \bar{u}}{\partial r^k}}$. It can be verified that \bar{v}^k satisfies

$$(5.24) \quad \begin{cases} \bar{L}(\bar{v}^k) = r^{k-2}(r^2 \bar{f}) & \text{in } S_\delta \\ \bar{v}^k|_{\partial S_\delta} = 0. \end{cases}$$

Then applying (5.23) to the problem (5.24) we obtain

$$\begin{aligned}
(5.25) \quad \|\bar{u}^k\|_{H_\beta^{2,2}(S_{\delta/2})} &\leq \|\bar{v}^k\|_{W_\beta^2(S_{\delta/2})} \\
&\leq \bar{C} \left[\|r^{k-2} \frac{\partial^k}{\partial r^k} (r^2 \bar{f})\|_{L_\beta(S_\delta)} + \|r^k \frac{\partial^k \bar{u}}{\partial r^k}\|_{H^1(S_\delta \setminus S_{\delta/2})} \right]
\end{aligned}$$

which implies that for $|\alpha| = k + 2$, $k > 0$, $0 \leq \alpha_2 \leq 2$

$$\begin{aligned}
(5.26) \quad \|r^{\alpha_1-2} \partial^{\alpha_1} \bar{u}\|_{L_\beta(S_{\delta/2})}^2 &\leq C \left[r^{k-2} \frac{\partial^k (r^2 \bar{f})}{\partial r^k} \|_{L_\beta(S_\delta)} + \|r^k \frac{\partial^k \bar{u}}{\partial r^k}\|_{H^1(S_\delta \setminus S_{\delta/2})} \right] \\
&\leq C \left[\|\bar{f}\|_{H_\beta^{k,0}(S_\delta)} + \|\bar{u}\|_{H^{k+1}(S_\delta \setminus S_{\delta/2})} \right].
\end{aligned}$$

For $\alpha_2 > 2$, for instance, $\alpha_2 = k + 2$, $k > 0$ we have from the equation (4.1)

$$\begin{aligned}
(5.27) \quad -\mu \frac{\partial^{k+2} u_r}{\partial \theta^{k+2}} &= r^2 \frac{\partial^k f_r}{\partial \theta^k} + (\lambda + 2\mu) \left[r^2 \frac{\partial^{k+2} u_r}{\partial r^2 \partial \theta^k} + r \frac{\partial^{k+1} u_r}{\partial r \partial \theta^k} - \frac{\partial^k u_r}{\partial \theta^k} \right] \\
&\quad + (\lambda + 3\mu) \frac{\partial^{k+1} u_\theta}{\partial \theta^{k+1}} + (\lambda + \mu) r \frac{\partial^{k+2} u_\theta}{\partial r \partial \theta^{k+1}},
\end{aligned}$$

$$\begin{aligned}
(5.28) \quad (\lambda + 2\mu) \frac{\partial^{k+2} u_\theta}{\partial \theta^{k+2}} &= r^2 \frac{\partial^k f_\theta}{\partial \theta^k} + \mu \left[r^2 \frac{\partial^{k+2} u_\theta}{\partial r^2 \partial \theta^k} + r \frac{\partial^{k+1} u_\theta}{\partial r \partial \theta^k} - \frac{\partial^k u_\theta}{\partial \theta^k} \right] \\
&\quad + (\lambda + 3\mu) \frac{\partial^{k+1} u_r}{\partial \theta^{k+1}} + (\lambda + \mu) r \frac{\partial^{k+2} u_r}{\partial r \partial \theta^{k+1}}.
\end{aligned}$$

(5.27) and (5.28) lead to

$$\begin{aligned}
(5.29) \quad \|r^{-2} \frac{\partial^{k+2} \bar{u}}{\partial \theta^{k+2}}\|_{L_\beta(S_{\delta/2})} &\leq \tilde{C} \left\{ \|r^{-2} \frac{\partial^k \bar{f}}{\partial \theta^k}\|_{L_\beta(S_\delta)} + \left\| \frac{\partial^{k+2} \bar{u}}{\partial r^2 \partial \theta^k} \right\|_{L_\beta(S_\delta)} \right. \\
&\quad \left. + \|r^{-1} \frac{\partial^{k+1} \bar{u}}{\partial r \partial \theta^k}\|_{L_\beta(S_\delta)} + \left\| r^{-2} \frac{\partial^k \bar{u}}{\partial \theta^k} \right\|_{L_\beta(S_\delta)} \right\}
\end{aligned}$$

$$+ \left\{ \|r^{-2} \frac{\partial^{k+1} \bar{u}}{\partial \theta^{k+1}}\|_{L_\beta(S_\delta)} + \|r^{-1} \frac{\partial^{k+2} \bar{u}}{\partial r \partial \theta^{k+1}}\|_{L_\beta(S_\delta)} \right\}$$

with $\tilde{C} = \max\left(\frac{1}{\mu}, 1 + \frac{3\lambda}{\mu}\right)$.

Suppose (5.21) holds with $\bar{G}^0 = 0$ for $0 \leq |\alpha| \leq k+2$ and $0 \leq \alpha_2 \leq k+1$. Then we have

$$(5.30) \quad \|r^{-2} \frac{\partial^{k+2} \bar{u}}{\partial \theta^{k+2}}\|_{L_\beta(S_{\delta/2})} \leq \tilde{C} \left[\|\bar{f}\|_{\mathcal{H}_\beta^{k,0}(S_\delta)} + \|\bar{u}\|_{\mathcal{H}^{k+1}(S_\delta \setminus S_{\delta/2})} \right]$$

which is (5.26) for $\alpha_2 = |\alpha| = k+2$. Actually we can prove (5.27) for each term of type $r^{\alpha_1-2} \partial^{\alpha} \bar{u}$ in exactly the same way. If $\bar{G}^0 \neq 0$, let $\bar{v} = \bar{u} - \bar{G}^0$, then applying the results above to \bar{v} we get (5.21) with absence of G^1 .

Next we shall prove (5.22) for case (i). Assume again that $G^0 = 0$, let $\bar{f} \in B_\beta^0(\Omega, C_{\bar{f}}, d_{\bar{f}})$ then

$$(5.31) \quad \begin{aligned} \|r^{k-2} \frac{\partial^k (r^2 \bar{f})}{\partial r^k}\|_{L_\beta(S_\delta)} &\leq \|r^k \frac{\partial^k \bar{f}}{\partial r^k}\|_{L_\beta(S_\delta)} + 2k \|r^{k-1} \frac{\partial^{k-1} \bar{f}}{\partial r^{k-1}}\|_{L_\beta(S_\delta)} \\ &\quad + k(k-1) \|r^{k-2} \frac{\partial^{k-2} \bar{f}}{\partial r^{k-2}}\|_{L_\beta(S_\delta)} \\ &\leq C_{\bar{f}} \left[d_{\bar{f}}^k + 2d_{\bar{f}}^{k-1} + d_{\bar{f}}^{k-2} \right] k! \\ &\leq 3C_{\bar{f}} d_{\bar{f}}^k k! \end{aligned}$$

Also due to (5.20) we have

$$(5.32) \quad \begin{aligned} \|r^k \frac{\partial^k \bar{u}}{\partial r^k}\|_{\mathcal{H}_\beta^1(S_\delta \setminus S_{\delta/2})} &\leq \left\| \frac{\partial^{k+1} \bar{u}}{\partial r^{k+1}} \right\|_{L^2(S_\delta \setminus S_{\delta/2})} + k \left\| \frac{\partial^k \bar{u}}{\partial r^k} \right\|_{L^2(S_\delta \setminus S_{\delta/2})} \\ &\quad + \left\| \frac{1}{r} \frac{\partial^k \bar{u}}{\partial r^k \partial \theta} \right\|_{L^2(S_\delta \setminus S_{\delta/2})} \end{aligned}$$

$$\leq C_0 \left[d_0^{k-1} (k-1)! + d_0^{k-2} k(k-2)! + d_0^{k-1} (k-1)! \right]$$

$$\leq 4C_0 d_0^{k-1} (k-1)!$$

Substituting (5.31)~(5.32) into (5.26) we get (5.22) for $|\alpha| = k+2$, $k > 0$, $0 \leq \alpha_2 \leq 2$ with $L = 4C(C_0 + C_{\bar{f}})$ and $D = \max(d_{\bar{f}}, d_{\bar{G}^0})$.

Now we shall prove (5.27) for $\alpha_2 = k+2$, $k > 0$. Suppose (5.22) holds for $0 \leq |\alpha| \leq k+2$ and $0 \leq \alpha_2 \leq k+1$, then by (5.29) we have

$$\begin{aligned} (5.33) \quad \|r^{-2} \frac{\partial^{k+2} \bar{u}}{\partial \theta^{k+2}}\|_{L_{\beta}(S_{\delta/2})} &\leq \tilde{C} \left\{ C_{\bar{f}} d_{\bar{f}}^k k! + LD^k P^{k-2} k! + LD^{k-1} P^{k-2} (k-1)! \right. \\ &\quad \left. + LD^{k-2} P^{k-2} (k-2)! + LD^{k-1} P^{k-1} (k-1)! + LD^k P^{k-1} k! \right\} \\ &\leq LD^k P^k k! \left\{ \frac{C_{\bar{f}}}{L} \left(\frac{d_{\bar{f}}}{DP} \right)^k + \frac{\tilde{C}}{P^2} + \frac{\tilde{C}}{DP^2 k} \right. \\ &\quad \left. + \frac{\tilde{C}}{P^2 D k(k-1)} + \frac{\tilde{C}}{DPk} + \frac{\tilde{C}}{P} \right\} \\ &\leq LD^k P^k k! \end{aligned}$$

where P and D are selected large enough, e.g. $P = 4\tilde{C}$ and $d_{\bar{f}}/DP < 1$.

Similarly, we can prove (5.22) for each term of type $r^{\alpha_1-2} D^{\alpha_2} \bar{u}$, $2 \leq \alpha_2 \leq |\alpha| = k+2$. For $\bar{G}^0 \neq 0$, let $\bar{w} = \bar{u} - \bar{G}^0$ and $\bar{f}^* = \bar{f} - \bar{L}(\bar{G}^0) \in B_{\beta}^0(\Omega, C_{\bar{f}^*}, d_{\bar{f}^*})$ with $C_{\bar{f}^*} = C_{\bar{f}} + (5\lambda + 11\mu) C_{\bar{G}^0}$ and $d_{\bar{f}^*} = \max(d_{\bar{f}}, d_{\bar{G}^0})$. Then applying the results above to \bar{w} and \bar{f} we get (5.22) in general (G^1 is absent), with $L = M_0(C_0 + C_{\bar{f}} + C_{\bar{G}^0})$ and $D = \max(d_{\bar{f}}, d_{\bar{G}^0}, d_0)$.

In case (ii): $\bar{v}^k = r^k \frac{\partial^k \bar{u}}{\partial r^k}$, $k > 2$ satisfies

$$(5.34) \quad \begin{cases} \bar{L}(\vec{v}^k) = r^{k-2} \frac{\partial^k(r^2 \bar{f})}{\partial r^k} & \text{in } S_\delta \\ (\sigma_{r\theta}, \sigma_\theta)^T \Big|_{\theta=0, \omega} = r^{k-1} \frac{\partial^k(r \bar{G}^1)}{\partial r^k} \Big|_{\theta=0, \omega} \end{cases}$$

Applying (5.23) to (5.34) we have

$$(5.35) \quad \|\vec{v}^k\|_{H_\beta^{2,2}(S_{\delta/2})} \leq \tilde{C} \left\{ \|r^{k-2} \frac{\partial^k(r^2 \bar{f})}{\partial r^k}\|_{L_\beta(S_\delta)} + \|r^{k-1} \frac{\partial^k(r \bar{G}^1)}{\partial r^k}\|_{\mathcal{H}_\beta^{1,1}(S_\delta)} + \|\vec{v}^k\|_{\mathcal{H}^1(S_\delta \setminus S_{\delta/2})} \right\}$$

which implies that for $|\alpha| = k + 2$, $k > 0$, $0 \leq \alpha_2 \leq 2$

$$(5.36) \quad \|r^{\alpha_1-2} \mathcal{D}^{\alpha_2} \bar{u}\|_{L_\beta(S_{\delta/2})} \leq \tilde{C} \left\{ \|r^{k-2} \frac{\partial^k(r^2 \bar{f})}{\partial r^k}\|_{L_\beta(S_\delta)} + \|r^{k-1} \frac{\partial^k(r \bar{G}^1)}{\partial r^k}\|_{\mathcal{H}_\beta^{1,1}(S_\delta)} + \|r^k \frac{\partial^k \bar{u}}{\partial r^k}\|_{\mathcal{H}^1(S_\delta \setminus S_{\delta/2})} \right. \\ \left. \leq C \left\{ \|\bar{f}\|_{\mathcal{H}_\beta^{k,0}(S_\delta)} + \|\bar{G}^1\|_{\mathcal{H}_\beta^{k+1,0}(S_\delta)} + \|\bar{u}\|_{\mathcal{H}^{k+1}(S_\delta \setminus S_{\delta/2})} \right\} \right\}.$$

For $2 < \alpha_2 \leq |k| = k + 2$, arguing as in case (i), we can conclude (5.22) with the absence of G^0 for $0 \leq \alpha_2 \leq |\alpha| = k + 2$.

If $\bar{f} \in \mathcal{B}_\beta^0(\Omega, C_{\bar{f}}, d_{\bar{f}})$, $\bar{G}^1 \in \mathcal{B}_\beta^1(\Omega, C_{\bar{G}^1}, d_{\bar{G}^1})$, then (5.32) holds, and

$$(5.37) \quad \|r^{k-1} \frac{\partial^{k-1}(r \bar{G}^1)}{\partial r^{k-1}}\|_{\mathcal{H}_\beta^{1,1}(S_\delta)} \leq \|r^k \frac{\partial^{k+1} \bar{G}^1}{\partial r^{k+1}}\|_{L_\beta(S_\delta)} + (2k+1) \|r^{k-1} \frac{\partial^k \bar{G}^1}{\partial r^k}\|_{L_\beta(S_\delta)} \\ + k(k-1) \|r^{k-2+\beta} \frac{\partial^{k-1} \bar{G}^1}{\partial r^{k-1}}\|_{L_\beta(S_\delta)} + \|r^{k-1} \frac{\partial^{k+1} \bar{G}^1}{\partial r^k \partial \theta}\|_{L_\beta(S_\delta)} \\ + k \|r^{k-2} \frac{\partial^k \bar{G}^1}{\partial r^{k-1} \partial \theta}\|_{L_\beta(S_\delta)}$$

$$\begin{aligned}
&\leq C_{\bar{G}^1} \left[d_{\bar{G}^1}^k k! + (2k+1) d_{\bar{G}^1}^{k-1} (k-1)! + d_{\bar{G}^1}^{k-2} k! \right. \\
&\quad \left. + d_{\bar{G}^1}^k k! + d_{\bar{G}^1}^{k-1} k! \right] \\
&\leq 6 C_{\bar{G}^1} d_{\bar{G}^1}^k k!
\end{aligned}$$

which together with (5.31)~(5.32) and (5.35)~(5.36) yield (5.22) for $|\alpha| = k + 2$, $k > 0$, $0 \leq \alpha_2 \leq 2$ with $L = 6\tilde{C}(C_0 + C_{\bar{f}} + C_{\bar{G}^1})$ and $D = \max(d_{\bar{f}}, d_{\bar{G}^1}, d_0)$. As arguing as in case (i) we can conclude (5.22) in general for case (ii).

The proof for case (iii) is similar to cases (i) and (ii). We will not repeat it.

Summarizing the analyses in each sector and interior we conclude that $\bar{u} \in \mathcal{H}_{\beta}^{k+2,2}(\Omega)$ and (5.22) holds if $\bar{f} \in \mathcal{H}_{\beta}^{k,0}(\Omega)$ and $\bar{G}^{\ell} \in \mathcal{H}_{\beta}^{k+2-\ell,2-\ell}(\Omega)$, $\ell = 0, 1$. Furthermore, if $\bar{f} \in \mathcal{B}_{\beta}^0(\Omega)$ and $\bar{G}^{\ell} \in \mathcal{B}_{\beta}^{2-\ell}(\Omega)$, $\ell = 0, 1$, then $\bar{u} \in \mathcal{B}_{\beta}^2(\Omega)$ with $C_{\bar{u}}$ and $d_{\bar{u}}$ satisfying

$$\begin{aligned}
C_{\bar{u}} &\leq M(C_{\bar{f}} + C_{\bar{G}^0} + C_{\bar{G}^1} + C_0) \\
d_{\bar{u}} &= \max\{d_{\bar{f}}, d_{\bar{G}^0}, d_{\bar{G}^1}, d_0\}
\end{aligned}$$

which together with (5.26a) imply

$$\begin{aligned}
C_{\bar{u}} &\leq M_c(C_{\bar{f}} + C_{\bar{G}^0} + C_{\bar{G}^1}) \\
d_{\bar{u}} &\leq M_d(d_{\bar{f}} + d_{\bar{G}^0} + d_{\bar{G}^1}).
\end{aligned}$$

By Theorem 2.1 and Lemma 5.2 $u \in B_{\beta}^2(\Omega, C_u, d_u)$ with some C_u and d_u satisfying (5.16).

Remark 5.3. If $\kappa_1^i = 1$ then β_i can be chosen any positive number. If $\kappa_1^i > 1$, the solution u may be smoother, for instance, $1 < \kappa_1^i < 2$, $1 \leq i \leq M$ and $f \in H_\beta^{k,1}(\Omega)$, (resp. $B_\beta^1(\Omega)$) $G^\ell \in H_\beta^{k+2-\ell,3-\ell}(\Omega)$ (resp. $B_\beta^{3-\ell}(\Omega)$) with $\beta_i \in (\kappa_1^i - 1, 1)$ and $k \geq 1$, then $u \in H_\beta^{k+2,3}(\Omega)$ (resp. $B_\beta^3(\Omega)$). In general, for $\kappa_1^i \in (n, n+1]$, $1 \leq i \leq M$. $u \in H_\beta^{k+2,2+n}(\Omega)$ (resp. $u \in B_\beta^{2+n}(\Omega)$) if f and G^ℓ are given in $H_\beta^{k,n}(\Omega)$ and $H_\beta^{k+2-\ell,n+2-\ell}(\Omega)$ (resp. $B_\beta^n(\Omega)$ and $B_\beta^{n+2-\ell}(\Omega)$).

Acknowledgement. The authors wish to thank Professor Bruce Kellogg for his helpful discussion during preparation of this paper.

APPENDIX

The transcendental equations (4.11) ~ (4.13) are given in Lemma 4.1 without proof. We shall derive these equations here. Consider the homogeneous problem

$$(A.1) \quad \mathcal{U}(\eta)\hat{\mathbf{u}} = [\hat{L}(D_\theta, \eta), \hat{B}(D_\theta, \eta)]\hat{\mathbf{u}} = 0$$

namely

$$(A.2) \quad \begin{aligned} & -\mu \frac{d^2 \hat{u}_\tau}{d\theta^2} + (2\mu + \lambda)(1 + \eta^2) \hat{u}_\tau + (i\eta(\mu + \lambda) + (3\mu + \lambda)) \frac{d\hat{u}_\theta}{d\theta} = 0 \\ & -((3\mu + \lambda) - i\eta(\mu + \lambda)) \frac{d\hat{u}_\tau}{d\theta} - (2\mu + \lambda) \frac{d^2 \hat{u}_\theta}{d\theta^2} + \mu(1 + \eta^2) \hat{u}_\theta = 0 \end{aligned} \quad \text{in } I = (0, \omega)$$

with one of the following boundary conditions:

$$(A.3a) \quad \hat{\mathbf{u}}|_{\theta=0, \omega} = 0 \quad (\text{Dirichlet condition})$$

$$(A.3b) \quad \hat{\mathbf{T}}|_{\theta=0, \omega} = (\hat{\sigma}_{\tau\theta}, \hat{\sigma}_{\theta\theta})^T|_{\theta=0, \omega} = 0 \quad (\text{Neumann condition})$$

$$(A.3c) \quad \hat{\mathbf{u}}|_{\theta=0} = 0, \quad \hat{\mathbf{T}}|_{\theta=\omega} = (\hat{\sigma}_{\tau\theta}, \hat{\sigma}_{\theta\theta})^T|_{\theta=\omega} = 0 \quad (\text{Mixed condition})$$

where $\hat{\sigma}_{\tau\theta}$ and $\hat{\sigma}_{\theta\theta}$ are given by (4.10).

Since the coefficients of the differential operator \hat{L} and boundary operator \hat{B} are constants, the solution $\hat{\mathbf{u}}$ can be written in $e^{b\theta}\mathbf{C} = e^{b\theta}(c_1, c_2)^T$, b and \mathbf{C} satisfy $A(b)\mathbf{C} = 0$ with

$$A(b) = \begin{bmatrix} -b^2\mu + (1 + \eta^2)(2\mu + \lambda) & (i\eta(\mu + \lambda) + (3\mu + \lambda))b \\ (i\eta(\mu + \lambda) - (3\mu + \lambda))b & -(2\mu + \lambda)b^2 + \mu(1 + \eta^2) \end{bmatrix}.$$

Then

$$(A.4) \quad \det A(b) = \mu(2\mu + \lambda)[(b^2 - (1 + \eta^2))^2 + 4b^2].$$

Let $\eta = iz$, by solving for b from $\det(A(b)) = 0$ we have

$$b_1 = i(z + 1), \quad b_2 = i(z + 1), \quad b_3 = i(z - 1), \quad b_4 = -i(z - 1).$$

For $z \neq 0, \pm 1$,

$$A(b_1) = (z + 1) \begin{bmatrix} (\lambda + 3\mu) - z(\mu + \lambda) & 1(-z(\mu + \lambda) + (3\mu + \lambda)) \\ -i(z(\mu + \lambda) + (3\mu + \lambda)) & (\lambda + 3\mu) + (\mu + \lambda)z \end{bmatrix}$$

the corresponding eigen vector $C^1 = (1, 1)^T$. Similarly we have $C^2 = (1, -1)^T$, $C^3 = (1, iH)^T$, $C^4 = (1, -iH)^T$ corresponding to b_2, b_3 and b_4 with

$$(A.5) \quad H = \frac{z(\mu + \lambda) + (3\mu + \lambda)}{z(\mu + \lambda) - (3\mu + \lambda)}.$$

Therefore, the solution of homogeneous problem has the following form:

$$(A.6) \quad \hat{u} = B_1 \begin{bmatrix} \cos(z + 1)\theta \\ -\sin(z + 1)\theta \end{bmatrix} + B_2 \begin{bmatrix} \sin(z + 1)\theta \\ \cos(z + 1)\theta \end{bmatrix} + B_3 \begin{bmatrix} \cos(z - 1)\theta \\ -H\sin(z - 1)\theta \end{bmatrix} \\ + B_4 \begin{bmatrix} \sin(z - 1)\theta \\ H\cos(z - 1)\theta \end{bmatrix}.$$

If $z = \pm 1$ (i.e., $\eta = \pm i$), $b_1 = 2i$, $b_2 = -2i$, $b_3 = b_4 = 0$, $A(b_3) = A(b_4)$ are null matrix. The corresponding eigen vectors are

$$C_1 = (1, 1)^T, C_2 = (1, -1)^T, C^3 = (1, 0)^T, C_4 = (0, 1)^T$$

and the solution of homogeneous problem has the form

$$\hat{u} = B_1 \begin{bmatrix} \cos 2\theta \\ -M\sin 2\theta \end{bmatrix} + B_2 \begin{bmatrix} \sin 2\theta \\ M\cos 2\theta \end{bmatrix} + B_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + B_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with

$$M = \begin{cases} 1 & \text{for } z = 1 \\ \frac{\mu}{2\mu + \lambda} & \text{for } z = -1 \end{cases}.$$

For $z = 0$, $b_1 = b_4 = i$, $b_2 = b_3 = -i$, the corresponding eigenvectors are

$$C_1 = (1, i)^T, C_2 = (1, -i)^T,$$

and the solution of the homogeneous problem has the form

$$(A.8) \quad \hat{u} = B_1 \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} + B_2 \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} + B_3 \begin{bmatrix} (3\mu + \lambda)\theta\cos\theta + \lambda\sin\theta \\ -(3\mu + \lambda)\theta\sin\theta - \mu\cos\theta \end{bmatrix} \\ + B_4 \begin{bmatrix} (3\mu + \lambda)\theta\sin\theta - \lambda\cos\theta \\ (3\mu + \lambda)\theta\cos\theta - \mu\sin\theta \end{bmatrix}.$$

$B = (B_1, B_2, B_3, B_4)^T$ would be determined according to the types of boundary conditions.

Lemma A.1. Let $\eta = iz$ be an eigenvalue of the operator $\mathcal{U}(\eta)$ for the Dirichlet problem (A.2) and (A.3a), then z satisfies the equation

$$(A.9) \quad \sin^2 z\omega = \left[\frac{z}{3-4\nu} \right]^2 \sin^2 \omega.$$

Proof. For the Dirichlet boundary condition $u|_{\theta=0, \omega} = 0$, if $z \neq 0, \pm 1$ due to (A.3a) and (A.6), B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & H \\ \cos(z-1)\omega & \sin(z+1)\omega & \cos(z+1)\omega & \sin(z+1)\omega \\ -\sin(z-1)\omega & \cos(z+1)\omega & -H\sin(z+1)\omega & H\cos(z+1)\omega \end{bmatrix}.$$

For the existence of non-trivial B it is sufficient and necessary that

$$(A.10) \quad \det(\Sigma) = (1 + H)^2 \sin^2 \omega - (1 - H)^2 \sin^2 z\omega = 0.$$

Note that $\frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}$ and

$$(A.11) \quad \frac{1+H}{1-H} = \frac{z(\mu+\lambda)}{3(\mu+\lambda)} = z \frac{(1+\frac{\lambda}{\mu})}{(3+\frac{\lambda}{\mu})} = \frac{z}{3-4\nu}$$

(A.10) and (A.11) yield (A.9).

If $z = \pm 1$, by (A.7) B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & M & 0 & 1 \\ \cos 2\omega & \sin 2\omega & 1 & 0 \\ -M \sin 2\omega & M \cos 2\omega & 0 & 1 \end{bmatrix}$$

and $\det(\Sigma) = 4M \sin^2 \omega$. Since $0 < \omega \leq 2\pi$, for $\omega = \pi, 2\pi$ the homogeneous problem has a non-trivial solution. But $z = \pm 1$ and $\omega = \pi, 2\pi$ satisfy (A.9). Hence $z = \pm 1$ are the zeroes of (A.9) for $\omega = \pi, 2\pi$, which are included in the equation (A.9).

If $z = 0$, by (A.5), B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & -\lambda \\ 0 & M & -\mu & 1 \\ \cos \omega & \sin \omega & (3\mu + \lambda)\omega \cos \omega + \lambda \sin \omega & (3\mu + \lambda)\omega \sin \omega - \lambda \cos \omega \\ -\sin \omega & \cos \omega & -(3\mu + \lambda)\omega \sin \omega - \mu \cos \omega & (3\mu + \lambda)\omega \cos \omega - \mu \sin \omega \end{bmatrix}$$

and $\det(\Sigma) = (3\mu + \lambda)^2 \omega^2 - (\lambda + \mu)^2 \sin^2 \omega > 0$ which implies $B = 0$. Hence zero is not an eigenvalue.

Lemma A.2. Let $\eta = iz$ be an eigenvalue of the operator $\mathcal{U}(\eta)$ for the Neumann problem (A.2) and (A.3b), then z satisfies the equation

$$(A.12) \quad \sin^2 z \omega = z^2 \sin \omega,$$

specially $\eta = 0$ is an eigenvalue with multiplicity of 2, the corresponding eigenvectors are $\bar{e}_1 = (\cos \theta, -\sin \theta)^T$ and $\bar{e}_2 = (\sin \theta, \cos \theta)^T$.

Proof. For Neumann boundary condition $\hat{\sigma}_{\tau\theta}|_{\theta=0, \omega} = \hat{\sigma}_{\theta\theta}|_{\theta=0, \omega} = 0$ we have by

(4.10)

$$\begin{aligned}
& \hat{\sigma}_{\tau\theta} = \mu \left[\frac{d\hat{u}_{\tau}}{d\theta} + (z-1)\hat{u}_{\theta} \right] = 0 \\
(A.13) \quad & \hat{\sigma}_{\theta\theta} = (2\mu+\lambda) \frac{\partial \hat{u}_{\theta}}{\partial \theta} + (2\mu+\lambda+\lambda z)\hat{u}_{\tau} = 0 .
\end{aligned}$$

For $z \neq 0, \pm 1$ by (A.13) and (A.6), B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 0 & 2\mu z & 0 & \mu(1+H)(z-1) \\ -2\mu z & 0 & A & 0 \\ -2\mu z \sin(z+1)\omega & 2\mu z \cos(z+1)\omega & -\mu(1+H)(z-1)\sin(z-1)\omega & \mu(1+H)(z-1)\cos(z-1)\omega \\ -2\mu z \cos(z+1)\omega & -2\mu z \sin(z+1)\omega & A \cos(z-1)\omega & A \sin(z-1)\omega \end{bmatrix}$$

where

$$(A.14) \quad A = (2\mu+\lambda) + \lambda z - H(z-1)(2\mu+\lambda) .$$

For the existence of non-trivial solutions it is necessary and sufficient that

$$\det(\Sigma) = -8z^2 \mu^2 \{ (A - \mu(1+H)(z-1))^2 \sin^2 \omega - (A + \mu(H+1)(z-1))^2 \sin^2 z\omega \} = 0 .$$

Hence we obtain the equation

$$\sin^2 z\omega = \left[\frac{A - \mu(H+1)(z-1)}{A + \mu(H+1)(z-1)} \right]^2 \sin^2 \omega .$$

By (A.5) and (A.14) we have

$$(A.15) \quad \frac{A - \mu(H+1)(z-1)}{A + \mu(H+1)(z-1)} = z$$

which implies (A.12) immediately.

For $z = 1$, by (A.13) and (A.7), B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 0 & 2\mu & 0 & 0 \\ -2\mu & 0 & 2(\mu+\lambda) & 0 \\ -2\mu\sin 2\omega & 2\mu\cos 2\omega & 0 & 0 \\ -2\mu\cos 2\omega & -2\mu\sin 2\omega & 2(\mu+\lambda) & 0 \end{bmatrix}$$

then $\det(\Sigma) = 0$, and $(0,0,0,1)^T$ is the corresponding solution of $\Sigma B = 0$. Accordingly $\hat{u} = (0,1)^T$, which represents a rotation around the origin.

For $z = -1$ by (A.13) and (A.7), B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 0 & 2\mu(1-M) & 0 & -2\mu \\ -2\mu(1-M)\sin 2\omega & -2\mu(1-M)\cos 2\omega & 0 & -2\mu \\ 0 & 0 & 2\mu & 0 \\ 0 & 0 & 2\mu & 0 \end{bmatrix}$$

then $\det(\Sigma) = 0$. Hence for $z = \pm 1$, the equation ΣB has a non-trivial solution. The null space corresponding to the eigenvalue $\eta = \pm i$ is not our interest here, and we will not elaborate on it further, but refer to [26].

For $z = 0$, by (A.5) and (A.8) B satisfies $\Sigma B = 0$ with

$$\Sigma = 2\mu(2\mu+\lambda) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \cos \omega & \sin \omega \\ 0 & 0 & -\sin \omega & \cos \omega \end{bmatrix}.$$

Obviously, $\det(\Sigma) = 0$, $\text{rank}(\Sigma) = 2$, and $(1,0,0,0)^T$ and $(0,1,0,0)^T$ are two linearly independent solutions of $\Sigma B = 0$. Accordingly the space of non-trivial solutions of homogeneous problem is spanned by

$$\bar{e}_1 = (\cos \theta, -\sin \theta)^T, \quad \bar{e}_2 = (\sin \theta, \cos \theta)^T$$

which represent the translation in x_1 and x_2 directions. \square

Lemma A.3. Let $\eta = iz$ be an eigenvalue of the operator $\mathcal{U}(\eta)$ for the mixed problem (A.2) and (A.3c), then z satisfies the equation

$$(A.16) \quad \sin^2 z\omega = \frac{4(1-\nu)^2}{3-4\nu} - \frac{z^2}{3-4\nu} \sin^2 \omega,$$

in the case that $(1+\frac{\lambda}{\mu})\cos 2\omega + 1 = 0$, $\eta = \pm i$ are the eigenvalues, and

(A.16) is satisfied.

Proof. For the mixed boundary condition we have $\bar{u}|_{\theta=0} = 0$ and $\bar{T}|_{\theta=\omega} = (\hat{\sigma}_{\tau\theta}, \hat{\sigma}_{\theta\theta})^T|_{\theta=\omega} = 0$. If $z \neq 0, \pm 1$ by (A.6) and (A.13), B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & H \\ -2\mu z \sin(z+1)\omega & 2\mu z \cos(z+1)\omega & -\mu(H+1)(z-1)\sin(z-1)\omega & \mu(H+1)(z-1)\cos(z-1)\omega \\ -2z \cos(z-1)\omega & -2\mu z \sin(z+1)\omega & A \cos(z-1)\omega & A \sin(z-1)\omega \end{bmatrix}$$

where A is given by (A.14). Then

$$\begin{aligned} \det(\Sigma) &= 4\mu^2 H z^2 - A\mu(1+H)(z-1) + 2\mu(HAz - \mu z(z-1)(1+H)) \\ &\quad - 2(1+H)z(A - (z-1)(1+H)\mu) \sin^2 \omega - 2(H-1)z\mu(A + \mu(z-1)(1+H)) \sin^2 z\omega. \end{aligned}$$

For the existence of non-trivial B , $\det(\Sigma) = 0$. Hence we have

$$\begin{aligned} (A.17) \quad \sin^2 z\omega &= \frac{4\mu H z^2 - A(1+H)(z-1) + 2z(HA - \mu(z-1)(1+H))}{2(1-H)z(A + (z-1)(1+H)\mu)} \\ &\quad - \frac{(H+1)(A - (z-1)(1+H)\mu)}{(H-1)(A + (z-1)(1+H)\mu)} \sin^2 \omega. \end{aligned}$$

Due to (A.11) and (A.14) we obtain

$$(A.18) \quad \frac{(H+1)(A-\mu(z-1)(H+1))}{(H-1)(A+\mu(z-1)(H+1))} = \frac{z^2}{3-4\nu}$$

and

$$(A.19) \quad \frac{4\mu H z^2 - A(H+1)(z-1) + 2z(HA - \mu(z-1)(H+1))}{2(H-1)z(A + (z-1)(1+H)\mu)} = \frac{4(1-\nu)^2}{3-4\nu}.$$

(A.17) - (A.19) yields (A.16).

For $z = 1$ by (A.13) and (A.7) B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -2\mu \sin 2\omega & 2\mu \cos 2\omega & 0 & 0 \\ -2\mu \cos 2\omega & -2\mu \sin 2\omega & 2(\mu+\lambda) & 0 \end{bmatrix}$$

then $\det(\Sigma) = 4\mu^2[(1+\frac{\lambda}{\mu})\cos 2\omega + 1]$. Similarly for $z = -1$

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & M & 0 & 1 \\ 0 & 0 & 2\mu & 0 \\ -2\mu(1-M)\sin 2\omega & 2\mu(1-M)\cos 2\omega & 0 & -2\mu \end{bmatrix}$$

and $\det(\Sigma) = \frac{4\mu^2}{2\mu+\lambda} [(1+\frac{\lambda}{\mu})\cos 2\omega + 1]$. Hence $z = \pm 1$ are eigenvalues if

$(1+\frac{\lambda}{\mu})\cos 2\omega + 1 = 0$, and also (A.16) is satisfied in the case that $z = \pm 1$ and $(1+\frac{1}{\mu})\cos 2\omega + 1 = 0$. Therefore the equation $\Sigma B = 0$ has a non-trivial solution in that case.

For $z = 0$ then, by (A.13) and (A.8) B satisfies $\Sigma B = 0$ with

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & -\lambda \\ 0 & 1 & -\mu & 0 \\ 0 & 0 & 2\mu(2\mu+\lambda)\cos\omega & 2\mu(2\mu+\lambda)\sin\omega \\ 0 & 0 & -2\mu(2\mu+\lambda)\sin\omega & 2\mu(2\mu+\lambda)\cos\omega \end{bmatrix}$$

then $\det(\Sigma) = 2\mu(2\mu+\lambda) > 0$. Hence $\eta = 0$ is not an eigenvalue of $\mathcal{U}(\eta)$. \square

Remark. The transcendental equations for the Dirichlet, Neumann and mixed boundary conditions were derived by using the argument of biharmonic function in [26] where displacement $u = (u_1, u_2)^T$ in Cartesian coordinates was used. Consequently the coefficients of the equation (A.1) are not constants. Therefore the simple argument of linear system of ordinary equations with constant coefficients is not valid. Nevertheless it shows that the transcendental equations are independent of choice of coordinates.

REFERENCES

- [1] Agranovich, M.S., Vishik, M.K.: Elliptic problems with a parameter and parabolic problems of general type, *Uspekhi Mat. Nauk.* 19:3 (1959), 626-727 (*Russian Math. Surveys* 19:3 (1964), 53-157).
- [2] Babuška, I., Guo, B.Q.: Regularity of the solution of elliptic problems with piecewise analytic data. Part 1: Boundary value problems for linear elliptic equation of second order, *SIAM J. Math. Anal.*, 19 (1988), 172-203.
- [3] Babuška, I., Guo, B.Q.: Regularity of the solution of elliptic problems with piecewise analytic data. Part 2: The trace spaces and application to boundary value problems with non-homogeneous conditions, *SIAM J. Math. Anal.* 20 (1989), 763-781.
- [4] Babuška, I., Guo, B.Q.: The h-p version of the finite element method with curved boundary, *SIAM J. Numer. Anal.* 24 (1988), 837-861.
- [5] Babuška, I., Guo, B.Q.: The h-p version of the finite element method for problems with non-homogeneous essential boundary condition, *Comp. Math. Appl. Mech. Eng.* 74 (1989), 1-28.
- [6] Babuška, I., Guo, B.Q., Osborn, J.E.: Regularity and numerical solution of eigen value problem with piecewise analytic data, *SIAM J. Numer. Anal.* 26 (1989), 1534-1560.
- [7] Babuška, I., Suri, M.: The p and h-p version of the finite element method. An Overview. *Compt. Methods Appl. Mech. Engrg.* 80, 1991, 5-26.
- [8] Babuška, I., Guo, B.Q., Stephen, E.P.: On the exponential convergence of the h-p version for boundary element Galerkin method on polygon, *Math. Meth. Appl. Sci.*, 12 (1990), 413-427.
- [9] Baouendi, M., Sjöstrand, J.: Analytic regularity for the Dirichlet problem in domains with conic singularities, *Ann. Scuola Norm. Pisa, Serie IV, Vol. IV 3* (1977), 515-530.
- [10] Blum, H., Rannacher, R.: On the boundary value problem of the biharmonic operators on domains with angular corners, *Math. Meth. in the Appl. Sci.* 2 (1980), 556-581.
- [11] Bolley, P., Dauge, M., Camus, J.: Regularite Gevrey pour le probleme de Dirichlet dans des domaines a singularites coniques, *Comm. in P.D.E.*, 10(4), 391-431, 1985.
- [12] Dauge, M.: Elliptic boundary value problems on corner domain, *Lecture Notes in Math* 1341, New York, Springer-Verlag, 1988.

- [13] Grivard, P.: Singularité en élasticité, Arch. for Rational Mechanics and Analysis, 107, 1989, 157-180.
- [14] Guo, B.Q., Babuška, I.: The h-p version of the finite element methods. Part 1: The basic approximation results, Comp. Mech. 1 (1986), 21-41, Part 2: General results and application, Comp. Mech. 1 (1986), 203-220.
- [15] Guo, B.Q., Babuska, I.: Regularity for interface problem in terms of countably normed spaces, to appear.
- [16] Goldberg, I.C., Krein, M.G.: Introduction to the theory of linear non-self-adjoint operators, American Mathematical Society, Providence, R.I., 1969.
- [17] Hlaváček, I., Nečas, J.: On inequalities of Korn's Type. I: Boundary value problems for elliptic systems of partial differential equations. II: Applications to linear elasticity. Arch rational Mech. Anal. 36, (1970), 305-311 and 312-334.
- [18] Kondratév, V. A.: Boundary value problem for elliptic equations in domain with cornic or angular points, Trans. Moscow Math. Soc., (1976), 227-313.
- [19] Kondratév, V.A.: Boundary value problems for parabolic equations in closed domains, Trans. Moscow Math. Soc., (1966), 450-504.
- [20] Kondratév, V.A., Oleinik, O.A.: Boundary value problems for partial differential equations in nonsmooth domains, Russian Math. Surveys, 38 (1983), 1-86.
- [21] Maz'ya, V.G., Plamenevskii, B.A.: Estimates in L_p and Hölder class and the Miranda-Agmon maximum principle for solutions of elliptic boundary problems in domains with singular points on the boundary, Amer. Math. Soc. Transl. (2), Vol. 123, 1984, 1-56.
- [22] Maz'ya, V.G., Plamenevskii, B.A.: On the coefficients in the asymptotic of solutions of elliptic boundary value problems in domain with conical points, Amer. Math. Soc. Transl. (2), Vol. 123, 1984, 57-88.
- [23] Morrey, C.B.: Multiple integrals in calculus of variations, Springer-Verlag, Berlin-Heidelberg-New York, 1966.
- [24] Melzer, H.K., Rannacher, R.: Spannungskonzentrationen in Eckpunkten der Kirchhoffschen Platte, Bauingenieur 55 (1980), 181-184.
- [25] Saada, A.S.: Elasticity, theory and applications, Pergamon Press Inc., New York, 1974.
- [26] Sändig, A.M., Sändig, R.: The regularity of boundary value problems for the Lamé equations in a polygonal domain, Restock Math., Kolloq. 36 (1989), 21-50.

- [27] Szabó, B.A.: The use of a priori estimates in engineering computations. Reliability in Computational Mechanics Ed. by J. T. Oden, 1989, 139-154.
- [28] Williams, M.L.: Surface stress singularities resulting from various boundary conditions in angular corners of plate under bending, Proc. First U.S. Nat. Congr. of Appl. Mech., 1951, 325-329.

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.
- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.
- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.
- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.
- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

Further information may be obtained from **Professor I. Babuška**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431.